THE VECTOR FIELD METHOD AND ITS APPLICATIONS TO NONLINEAR EVOLUTION EQUATIONS

By

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ABSTRACT

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The vector field method was introduced in the 1980s by Sergiu Klainerman to analyze the decay properties of the linear wave equation. Since its historical debut, the vector field method has been at the forefront of several breakthrough results including the global stability of Minkowski space, the dynamical formation of black holes, and shock formation in $3D$ compressible fluids.

This work showcases how the vector field method can be used in a systematic way to derive a priori estimates for nonlinear evolution equations. For nonlinear dispersive equations, these estimates can be married to the decay properties enjoyed by the solutions to derive quantitative asymptotics. This is done in this work through the lens of three concrete problems: a nonlocal kinetic model, the wave maps equation, and the relativistic membrane equation. For the kinetic model, the vector field method is paired with dispersive decay properties of the spatial density to prove global wellposedness of small data. This can be interpreted physically as "stability" of the trivial solution. For the wave maps equation, a stability result is proven for a "non-trivial" ODE geodesic solution. For the relativistic membrane equation, the vector field method is used to prove stability of large simple-traveling-waves. For the wave map and membrane equations, we intimately use several structural properties known as null conditions that preclude singular behavior.

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To Britta and Little Leo, for your endless love and support.
A mis padres, por sus sacrificios.
To Esteban, I hope you find a righteous future.
To the families of those who were wrongfully taken from them. **Black lives matter.**
To all of the lives lost due to COVID-19.

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KEY TO SYMBOLS

$\mathfrak{m}$	Minkowski metric with signature $(-1, 1, \ldots, 1)$.
$\mathbb{R}^{1,d}$	$(1+d)$-dimensional Minkowski space equipped with $\mathfrak{m}$. We will alternate freely between the notations $\mathbb{R}^{1,d}$ and $\mathbb{R}^{1+d}$. We denote the standard Euclidean coordinates on $\mathbb{R}^{1,d}$ by $\{x^\mu\}_{\mu\in\{0,\ldots,d\}}$. We will often write t for the time coordinate x^0.
$\Box_g$	Covariant wave operator on a pseudo-Riemannian manifold (M, g). In any local coordinate system, for a scalar function f, $$\Box_g f = \lvert\det g\rvert^{-1/2}\partial_\mu(\lvert\det g\rvert^{1/2} g^{\mu\nu}\partial_\nu f).$$
$\Box$, $\Box_{\mathfrak{m}}$	The d'Alembertian on $\mathbb{R}^{1,d}$. In rectangular coordinates, $\Box = -\partial_t^2 + \Delta_x$.
$\mathcal{L}^p_\alpha$	Weighted Lebesgue spaces. See notation in Section 2.1.
$\mathring{\mathcal{W}}^{k,p}_\alpha, \mathcal{W}^{k,p}_\alpha$	Weighted homogeneous and inhomogeneous and Sobolev spaces. See notation in Section 2.1.
∇^M	Levi–Civita connection on a pseudo-Riemannian manifold (M, g).
$\mathcal{L}_X T$	Lie differentiation of a scalar or tensor T in the direction of the vector field X.
$\widehat{u}$	Fourier transform of a function, scaled by $\widehat{u}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x$.
$\mathscr{S}(\mathbb{R}^d)$	Schwartz space of functions.
$\mathring{g}$	Linear part of the dynamical metric; see (4.2.11) in §4.2.2.
g	Dynamical metric; see (4.2.9) in §4.2.2.
Υ	planewave background; see (4.2.1) in §4.2.1.
$u, \underline{u}, \hat{x}$	Null coordinates adapted to planewave background; see (4.2.2) and the discussion after (4.2.5) in §4.2.1.
y^i	Rectangular coordinates adapted to planewave background; see (4.3.8) in §4.3.2.
$\mathcal{I}^+$	Forward light cone; see start of §4.3.1.
τ, ρ, Σ_τ	Hyperboloidal foliation and related parameters; see (4.3.1) and Notation 4.3.1 in §4.3.1.
T, L^i	Vector fields; see (4.3.5), (4.3.6), and (4.3.7) in §4.3.1.
$\mathcal{Q}$	Stress-energy tensors; see (4.3.10) in §4.3.1.
$\mathcal{E}$	Background energy integrals; see (4.3.13) and (4.3.14) in §**??**.
$\mathfrak{W}_*$	Weight functions; see Definition 4.3.5 in §4.3.2.

$\mathscr{W}_*$	Elements of $\mathfrak{W}_*$; see Notation 4.5.2 in §4.5.1.
$\mathscr{P}_*$	planewave like weights; see Notation 4.5.2 in §4.5.1.
$\mathfrak{A}_*$	Weighted commutator algebra; see discussion surrounding Proposition 4.3.8 in §4.3.2.
$\mathfrak{B}_*^{*,*}$	Weighted differential operators; see discussion surrounding (4.3.9) in §4.3.2, as well as Definition 4.3.10.
$\mathscr{B}_*^{*,*}$	Elements of $\mathfrak{B}_*^{*,*}$; see Notation 4.5.4 in §4.5.1.
$\mathscr{G}$	Smooth functions representing bounded terms; see Notation 4.5.6 in §4.5.1.

CHAPTER 1

INTRODUCTION

1.1 Prologue

Decay properties of solutions to nonlinear dispersive equations play a central role in their long-time behavior in the small data regime. This regime is important as it can be interpreted as a kind of "stability" of the trivial solution. One expects to be able to solve a nonlinear equation for a small amount of time so long as the nonlinearities are sufficiently "weak", which is typically a consequence of the small size of the initial data, and hence the dynamics are essentially linear. A global-in-time solution can then be constructed so long as the nonlinearities can be shown to decay sufficiently fast.

Classically, pointwise L^1–L^∞ decay for solutions to linear dispersive equations (such as Schrödinger, Airy, wave) is established by either estimating directly explicit representation formulas for their fundamental solutions, or by oscillatory integral techniques applied to the Fourier representations. These methods are either infeasible or impractical when the equations are perturbed in a way that changes the principal part (i.e. the part with the highest derivatives).

In 1985, Sergiu Klainerman developed a technique [Kla85b], now referred to as the vector field method (VFM), to systematically analyze the decay properties of the linear wave equation. The appeal of Klainerman's technique is that it is both a robust method that is well suited to handle small perturbations of the principal part of linear waves, and that the general mechanism is applicable to settings of other dispersive equations [Smu16, FJS17, Won18a]. The general strategy of the vector field method for *linear* equations can be summarized as follows:

I: Analyze the inherent symmetries of the equation by looking for commuting vector fields X that are *weighted* in time. That is, if ϕ solves the equation so does $X\phi$.

II: Analyze the inherent symmetries of the equation through its conservation laws.

III: Develop space-time weighted Sobolev inequalities that combine the previous two points to get lower derivative pointwise estimates from higher derivative integral norms. The temporal decay will be a consequence of these L^∞ estimates and from the temporal weight of X.

Even though exact conservation laws are typically not present for nonlinear equations, one can derive "approximate conservation laws" via *energy estimates*. In the spirit of Duhamel's formula, these estimates control an appropriately defined "size" of the solution at a later time by the initial size and an integral measuring interactions between the linear flow and the nonlinearities. In order to counteract the positive feedback of these nonlinearities, which would make these integrals diverge, we rely on both the linear decay extracted from the vector field method *and* smallness of the initial data to prevent the nonlinearities from growing too large.

This dissertation showcases the balancing act between growth and decay in the concrete settings of some wave equations which arise in Lagrangian field theories: the wave maps equation on Minkowski space and the relativistic membrane equation (see [SS98, Kri07, Hop13] for an overview of the physical and mathematical history). This will be done by using the vector field method as the technical skeleton to study decay, regularity, stability properties of solutions to these equations.

1.1.1 Brief outline of the present work

This introductory chapter is split into two halves. In Section 1.2, we provide a hands-on exposition of the VFM by using it to analyzing a toy model of nonlinear kinetic theory. The reader should keep in mind that the analytical themes in Section 1.2 will be the technical skeleton for the rest of the dissertation, see Subsection 1.2.6 for more details. In Section 1.3, we introduce the main results of this dissertation and explain how they pertain to a novel research program inspired by recent work. We highlight the important role the decay estimates, which are derived using the vector field method, have in the analysis.

Chapter 2 serves to establish necessary results for the linear wave equation. In Section 2.1 we begin by setting up the geometric formalism of a *hyperboloidal foliation* of Minkowski space. The presentation is adapted from the foundational work of Lefloch and Ma [LM14] and the recent work of [Won17b]. This hyperboloidal viewpoint is taken to exploit the fact that the symmetries of Minkowski space are *Lorentzian* and so "space" and "time" are on equal footing, as opposed to Galilean symmetries of Newtonian mechanics. The main results of this section are space-time weighted versions of the Gagliardo-Nirenberg-Sobolev (GNS) interpolation inequalities adapted to this hyperboloidal foliation, and are based on original joint work with Wong [AW19b]. These interpolation inequalities allow one to save derivatives in Lebesgue product estimates compared to the Morrey L^2–L^∞ Sobolev embeddings. In Section 2.2 we marry the vector field method and these weighted Sobolev estimates to derive *a priori estimates* for solutions to the linear wave and Klein–Gordon equations.

In Chapter 3, we show that a totally geodesic map from Minkowski space into a spaceform, under a sign condition, is globally nonlinearly stable as a solution to the wave maps equation under sufficiently small compactly supported perturbations. This is based on original joint work with Chen [AC19]. We begin by constructing the perturbation as a section of the normal bundle of the totally geodesic background. Using the spaceform assumption on the target, we prove that the equations of motion for the perturbation reduce to a semilinear system of wave-Klein–Gordon equations. Global existence and uniform decay estimates are proved for this perturbation using the vector field method and the GNS interpolation estimates of Chapter 2.

The main result of Chapter 4 is to show that planar traveling wave solutions to the membrane equation are globally nonlinearly stable under sufficiently small compactly supported perturbations. This is based on original joint work with Wong [AW19a]. We begin by writing the perturbation equations in a convenient gauge which reveals that the perturbed system can be described by a quasilinear perturbation of the linear wave equation on Minkowski space, with the background solution only appearing as coefficients of the nonlinearity. In view of this special geometric feature, we do not need to develop special methods to perform the linear analysis and can in large rely on the vector field method approach introduced for the linear wave equation in Chapter 2. The focus is almost entirely on the nonlinearity, with the main difficulty arising precisely from the non-decaying background contribution.

1.2 Prelude to the vector field method

In this section, we illustrate the vector field method through the lens of a concrete example. The notions of local and global wellposedness are also introduced and made rigorous. Moreover, the analysis of this concrete example also serves to introduce the *bootstrap principle*, a fundamental technical tool used in our analysis of the nonlinear kinetic model of this section, wave map equations of Chapter 3 and the membrane equation of Chapter 4. Essentially, it allows one to assume "for free" that the solution in question already obeys some quantitative bound which allows one to to prove, with the estimates afforded by the vector field method, another *stronger* bound (to avoid circularity). See Appendix A.1 for an abstract formulation of the bootstrap principle and the proof of Theorem 1.2.3 for a concrete application of the bootstrap principle. The reader should keep in mind that the analysis of Chapters 2 – 4 will encompass the techniques and methods mentioned in this prelude as a template.

1.2.1 The model

The Vlasov equation describes how a *distribution* of particles evolves in time. Newton's laws of motion assert that its evolution is determined by its initial position and velocity.

Consequently, we define the classical phase space $\mathbb{R}^d_x \times \mathbb{R}^d_v$ as the space of all possible configurations. A distribution of particles is then a function

$$\rho : \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v \to \overline{\mathbb{R}_+}$$

which measures the density of the particles at a particular position x, velocity v, and time t. The Vlasov equation then manifests itself as Newton's first law of motion under the simplification that particles don't interact:

$$\partial_t \rho(t,x,v) + v \cdot \partial_x \rho(t,x,v) = 0.$$

In this chapter we prove the existence of unique global-in-time solutions for the initial value problem

$$\begin{cases} \partial_t \rho + v \cdot \partial_x \rho = \int_{\mathbb{R}^d} \rho(t,x,v-v')\rho(t,x,v+v')\, \mathrm{d}v' \\ \rho(0,x,v) = \rho_0(x,v) \end{cases} \tag{1.2.1}$$

in the small data regime whenever $d \geq 2$ and small data exponential time existence for $d = 1$. The nonlinearity is not chosen to be physically meaningful; rather it was chosen for ease of exposition to aid in illustrating the vector field method through the lens of a concrete example. Despite this, the nonlocal term is sensible in that it averages particle interactions with different speeds (at the same point in space-time) over all possible velocities. In this way this model can be thought of as a poor man's Boltzmann. Moreover, the method of characteristics shows that a naive quadratic nonlinearity such as $\partial_t \rho + v \cdot \partial_x \rho = \rho^2$ would blow up in finite time for any $\rho_0 \in C_c^\infty(\mathbb{R}^{2d})$. Finally, similar nonlinearities are recovered after taking an appropriate transform of some well known equations. For example, it is known that the nonlinear Schrödinger equation

$$i\partial_t u - \Delta_x u = |u|^\sigma u \tag{1.2.2}$$

has small data global-in-time existence when $d \geq 3$ and $\sigma \in \left(1, \frac{4}{d-2}\right)$. The Fourier analogue of (1.2.2) is precisely

$$i\partial_t \widehat{u} + |\xi|^2 \widehat{u} = \frac{1}{(2\pi)^{\frac{d}{2}}} \widehat{|u|^\sigma} * \widehat{u}.$$

This shows that the nonlinearity in (1.2.1) is of a similar type as the one found in the Fourier ODE analogue of the nonlinear Schrödinger equation.

1.2.2 Notational conventions

Several notational conventions used in this subsection are now established. For the classical phase space we introduce the shorthand notation $\mathbb{R}^{2d}_{x,v} \stackrel{\text{def}}{=} \mathbb{R}^d_x \times \mathbb{R}^d_v$. The Vlasov operator will be denoted as

$$X \stackrel{\text{def}}{=} \partial_t + v \cdot \partial_x. \tag{1.2.3}$$

It is convenient to define the bilinear form

$$F(\rho_1, \rho_2) \stackrel{\text{def}}{=} F(\rho_1, \rho_2)(t, x, v) = \int_{\mathbb{R}^d_{v'}} \rho_1(t, x, v - v')\rho_2(t, x, v + v') \,\mathrm{d}v'. \tag{1.2.4}$$

for functions $\rho_1, \rho_2 : \mathbb{R}_t \times \mathbb{R}^{2d}_{x,v} \to \overline{\mathbb{R}_+}$. With this at hand, our nonlocal Cauchy problem (1.2.1) can be written as

$$\begin{cases} X\rho = F(\rho, \rho) \\ \rho|_{t=0} = \rho_0. \end{cases}$$

Let $\rho \in C^\infty_c(\mathbb{R}^{2d}_{x,v})$ and define the norms

$$\|\rho\|_{L^1_x L^1_v} \stackrel{\text{def}}{=} \int_{\mathbb{R}^{2d}_{x,v}} |\rho(x, v)| \,\mathrm{d}x\mathrm{d}v$$

$$\|\rho\|_{W^{k,1}_x L^1_v} \stackrel{\text{def}}{=} \sum_{|\alpha| \le k} \iint_{\mathbb{R}^{2d}_{x,v}} |\partial^\alpha_x \rho(x, v)| \,\mathrm{d}x\mathrm{d}v.$$

The phase-space Lebesgue and Sobolev spaces $L^1_x L^1_v$, $W^{k,1}_x L^1_v$ will be the completion of $C^\infty_c(\mathbb{R}^{2d}_{x,v})$ under the respective norms. The space $L^\infty_x L^1_v$ will be the space of Lebesgue measurable functions on $\mathbb{R}^{2d}_{x,v}$ such that

$$\|\rho\|_{L^\infty_x L^1_v} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{x \in \mathbb{R}^d_x} \|\rho(x, \cdot)\|_{L^1(\mathbb{R}^d_v)} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d_x} \int_{\mathbb{R}^d_v} |\rho(x, v)| \,\mathrm{d}v < \infty.$$

Finally, we also introduce the shorthand

$$(Y^k, \|\cdot\|_{Y^k}) \overset{\text{def}}{=} \left(W_x^k L_v^1, \|\cdot\|_{W_x^{k,1} L_v^1} \right)$$

and the ball centered at zero of radius R as $B^{Y^k}(R) := \{\rho \in Y^k \mid \|\rho\|_{Y^k} < R\}$. Note that $Y^0 = L_x^1 L_v^1$. In the sequel, when there is no ambiguity on k, we simply denote $\|\cdot\|_{Y^k} = \|\cdot\|_Y$.

Remark 1.2.1. In this notation, for functions $\rho : \mathbb{R}_t \times \mathbb{R}_{x,v}^{2d} \to \mathbb{R}$, the Sobolev inequality (A.1.2) reads

$$\|\rho(t, —)\|_{L_x^\infty L_v^1} \lesssim_{k,d} \|\rho(t, —)\|_{Y^k} \tag{1.2.5}$$

for any $k \geq d$.

1.2.3 Methodology

An initial value problem is said to be *wellposed* in the sense of Hadamard if there exists a unique solution with continuous dependence on the initial data. Wellposedness results are twofold; local wellposedness (LWP) and global wellposedness (GWP). Typical results for the former are that for arbitrary initial data, there exists a positive time of existence. Results for the latter are those in which the time of existence is infinite.

The main approach to prove LWP results for nonlinear evolution equations is to construct approximating solutions that, in a limit, converge to a true solution. One such method is Picard iteration which generalizes the proof of the Picard-Lindelöf theorem. Broadly speaking, one *linearizes* the problem by solving iteratively the corresponding linear equation where the nonlinearity uses the previous iterate. For semilinear equations this can be formulated as a contraction mapping argument because the equation is linear on the highest derivative terms and their coefficients do not depend on lower derivative terms. For quasilinear equations this is not the case because the principal term can have coefficients which depend on the solution itself, and as a consequence, convergence issues are more delicate, see [Hör97, Sog08].

The mechanism for the GWP results of this dissertation is the so called *breakdown criterion*. Roughly stated, a solution exists globally if and only if its "size" (as measured by an appropriate norm) does not diverge to infinity in finite time. The intuition behind the breakdown criterion is that a singularity can form if the solution becomes too large to iterate the approximating solutions from the LWP result indefinitely. The fundamental tool which provides control on the solution at later times are the (approximate)-*conservation* laws[1]. They control the solution by an integral that measures the interactions between the linear and nonlinear dynamics. In order to show that these interactions are integrable, and hence the solution is global as per the breakdown criterion, we rely wholeheartedly on the decay estimates which serve to dampen their feedback.

The local and global wellposedness results for the nonlocal Vlasov model (1.2.1) are:

Theorem 1.2.2 (Local wellposedness). *For any $R > 0$ there exists a time $T > 0$ such that, for any $\rho_0 \in B^{Y^{2d+2}}(R) \subset Y^{2d+2}$, there exists a unique $\rho \in C^0([0,T];Y^{2d+2})$ solving (1.2.1). In addition, solutions have a Lipschitz dependence on initial data, i.e. there exists a universal constant $C > 0$ such that, for any $\rho_0, \rho_0' \in B^{Y^{2d+2}}(R)$, their respective solutions satisfy*

$$\|\rho - \rho'\|_{C^0([0,T];Y^{2d+2})} \leq C\|\rho_0 - \rho_0'\|_{Y^{2d+2}}.$$

Theorem 1.2.3 (Global existence). *Let $\rho_0 \in Y^{2d+2}$ and consider the initial value problem*

$$\begin{cases} X\rho = F(\rho,\rho) \\ \rho(0,x,v) = \epsilon\rho_0(x,v). \end{cases} \tag{1.1$_\epsilon$}$$

Let $T_*^\epsilon := \sup\{T > 0 \mid \exists \rho \in C^0([0,T];Y^{2d+2}) \cap C^1([0,T];Y^{2d+1})$ solving $(1.1_\epsilon)\}$. Then

(a) if $d \geq 2$ then there exists an $\epsilon_0 > 0$ depending on at most $2d+1$ derivatives of ρ_0 such that $T_*^\epsilon = \infty$ for all $\epsilon < \epsilon_0$;

[1] In the wave equation setting, these are the energy estimates.

(b) *if* $d = 1$ *then there exist constants* $B, \epsilon_0 > 0$ *depending on at most 3 derivatives of* ρ_0 *such that*

$$T_*^\epsilon \geq \sinh\left(\frac{B}{\epsilon}\right)$$

for all $\epsilon < \epsilon_0$.

1.2.4 Approximate-conservation laws

We begin by examining the inhomogeneous linear Vlasov equation

$$\begin{cases} \partial_t \rho + v \cdot \partial_x \rho = F(t,x,v) \\ \rho(0,x,v) = \rho_0(x,v). \end{cases} \tag{1.2.6}$$

Duhamel's principle explicitly gives the solution as

$$\rho(t,x,v) = \rho_0(x - tv, v) + \int_0^t F(s, x + (s-t)v, v)\, \mathrm{d}s.$$

From this one immediately sees that if $\rho_0 \in \mathscr{S}(\mathbb{R}^{2d})$ and $F \in C^\infty([0,T];\mathscr{S}(\mathbb{R}^{2d}))$, then $\rho \in C^\infty([0,T];\mathscr{S}(\mathbb{R}^{2d}))$.

The vector field method described in the next subsection has its origins rooted in commutation relations such as

$$[X, \partial_x^\alpha] = 0, \tag{1.2.7}$$

where X is as in (1.2.3) and α is any multi-index. This fact can be used to prove the following estimate.

Lemma 1.2.4. *Let* $\rho_0 \in \mathscr{S}(\mathbb{R}^{2d})$, $F \in C^\infty([0,T];\mathscr{S}(\mathbb{R}^{2d}))$ *and suppose that* ρ *is a solution to (1.2.6). Then*

$$\|\rho(t,—)\|_{Y^s} \leq \|\rho(0,—)\|_{Y^s} + \int_0^t \|F(\tau,—)\|_{Y^s}\, \mathrm{d}\tau. \tag{1.2.8}$$

for any $s \in \mathbb{N} \cup \{0\}$.

Proof. Let $G : \mathbb{R} \to \mathbb{R}$ be any smooth function. Fix a multi-index α and use Stoke's theorem, the Schwarz assumption, and (1.2.7) to compute

$$\begin{aligned}
\partial_t \int\limits_{\mathbb{R}^{2d}} G(\partial_x^\alpha \rho(t,x,v))\,\mathrm{d}x\mathrm{d}v &= \partial_t \int\limits_{\mathbb{R}^{2d}} G(\partial_x^\alpha \rho(t,x,v))\,\mathrm{d}x\mathrm{d}v + \underbrace{\int\limits_{\mathbb{R}^{2d}} v \cdot \partial_x (G(\partial_x^\alpha \rho(t,x,v)))\,\mathrm{d}x\mathrm{d}v}_{=0} \\
&= \int\limits_{\mathbb{R}^{2d}} X(G(\partial_x^\alpha \rho(t,x,v)))\,\mathrm{d}x\mathrm{d}v \\
&= \int\limits_{\mathbb{R}^{2d}} G'(\partial_x^\alpha \rho(t,x,v)) X \partial_x^\alpha \rho(t,x,v)\,\mathrm{d}x\mathrm{d}v \\
&= \int\limits_{\mathbb{R}^{2d}} G'(\partial_x^\alpha \rho(t,x,v)) \partial_x^\alpha X \rho(t,x,v)\,\mathrm{d}x\mathrm{d}v.
\end{aligned}$$

Integrate this equality from 0 to t to find

$$\int\limits_{\mathbb{R}^{2d}} G(\partial_x^\alpha \rho(t,x,v))\,\mathrm{d}x\mathrm{d}v = \int\limits_{\mathbb{R}^{2d}} G(\partial_x^\alpha \rho(0,x,v))\,\mathrm{d}x\mathrm{d}v + \int\limits_0^t \int\limits_{\mathbb{R}^{2d}} G'(\partial_x^\alpha \rho) \partial_x^\alpha X \rho(\tau,x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}\tau.$$

Approximate $x \mapsto |x|$ by the smooth function $G : x \mapsto \sqrt{\epsilon + |x|^2}$ as $\epsilon \searrow 0$ to find

$$\begin{aligned}
\int\limits_{\mathbb{R}^{2d}} |\partial_x^\alpha \rho(t,x,v)|\,\mathrm{d}x\mathrm{d}v &= \int\limits_{\mathbb{R}^{2d}} |\partial_x^\alpha \rho(0,x,v)|\,\mathrm{d}x\mathrm{d}v + \int\limits_0^t \int\limits_{\mathbb{R}^{2d}} sgn(\partial_x^\alpha \rho) \partial_x^\alpha X \rho(\tau,x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}\tau \\
&\leq \int\limits_{\mathbb{R}^{2d}} |\partial_x^\alpha \rho(0,x,v)|\,\mathrm{d}x\mathrm{d}v + \int\limits_0^t \int\limits_{\mathbb{R}^{2d}} |\partial_x^\alpha X \rho(\tau,x,v)|\,\mathrm{d}x\mathrm{d}v\mathrm{d}\tau.
\end{aligned}$$

Summing up this inequality over all multi-indices of length s yields the claim. □

Remark 1.2.5. If $F \equiv 0$ in (1.2.6), then (1.2.8) is the usual conservation of mass result

$$\|\rho(t,\text{—})\|_{Y^s} = \|\rho(0,\text{—})\|_{Y^s}.$$

1.2.5 Wellposedness

In this final section we provide proofs for the wellposedness results of Theorems 1.2.2 and (1.2.3). The proof of LWP will be established via a Picard approximation argument.

We then establish the breakdown criterion for our non-local Vlasov model and exploit it by using the vector field method to prove global existence.

Proof of Theorem 1.2.2. Assume the initial data ρ_0 is compactly supported in phase-space; the full statement of Theorem 1.2.2 can be recovered by a density argument because $C_c^\infty(\mathbb{R}^{2d})$ is dense in Y^{2d+2} (which will be denoted as $\mathcal{Y}$ in this proof). Let $\rho^{(-1)} \equiv 0$ and define $\{\rho^{(j)}\}_{j=0}^{\infty}$ inductively as solutions to

$$\begin{cases} X\rho^{(j)} = F(\rho^{(j-1)}, \rho^{(j-1)}), \\ \rho^{(j)}(0,x,v) = \rho_0(x,v). \end{cases} \tag{1.2.9}$$

We begin by showing that each $\rho^{(j)} \in C^\infty([0,T]; C_c^\infty(\mathbb{R}^{2d}))$ with an induction argument. Note that compact support of ρ_0 implies the existence of constants $R_x, R_v > 0$ such that $\rho_0(x,v) = 0$ whenever $|x| > R_x$ or $|v| > R_v$. The first iterate $\rho^{(0)}(t,x,v)$ solves the homogeneous Vlasov equation $X\rho^{(0)} = 0$ and so the explicit solution is given to be $\rho^{(0)} = \rho_0(x - tv, v)$ by the method of characteristics. An immediate consequence is

$$\rho^{(0)}(t,x,v) \in C^\infty([0,T]; C_c^\infty(\mathbb{R}^{2d})),$$

and specifically, $\rho^{(0)}(t,x,v) = 0$ whenever $|v| > R_v$. Assume the induction hypothesis $\rho^{(j-1)} \in C^\infty([0,T]; C_c^\infty(\mathbb{R}^{2d}))$ with $\rho^{(j-1)}(t,x,v) = 0$ whenever $|v| > R_v$. The initial value problem (1.2.9) is solved by Duhamel's formula

$$\rho^{(j)}(t,x,v) = \rho_0(x - vt, v) + \int_0^t F(\rho^{(j-1)}, \rho^{(j-1)})(s, x + (s-t)v, v)\, \mathrm{d}s.$$

It is clear that $\rho^{(j)}$ is compactly supported in the spatial variables from the induction hypothesis. However, it is not clear that this is the case for the momentum variables because $\rho^{(j)}$ depends on the non-local term F which is an integral over all of $\mathbb{R}^d_{v'}$. The induction hypothesis states that $\rho^{(j-1)}(s, x + (s-t)v, v - v') = 0$ whenever $|v - v'| > R_v$ and $\rho^{(j-1)}(s, x + (s-t)v, v + v') = 0$ whenever $|v + v'| > R_v$. The Duhamel expression is then equivalent to

$$\rho^{(j)}(t,x,v) = \rho_0(x-vt,v) + \int_0^t \int_{\{|v-v'|\le R_v\}\cap\{|v+v'|\le R_v\}} \rho^{(j-1)}(s,x+(s-t)v,v-v') \times \rho^{(j-1)}(s,x+(s-t)v,v+v')\,\mathrm{d}v'ds.$$

The intersection $\{|v-v'| \le R_v\} \cap \{|v+v'| \le R_v\}$ is empty whenever $|v| > R_v$, and this concludes the proof of the induction.

Denote the initial mass by $A_0 \stackrel{\text{def}}{=} \|\rho_0\|_{\mathcal{Y}}$ and the mass at time t as

$$A_j(t) \stackrel{\text{def}}{=} \sum_{|\alpha|\le 2d+2} \|\partial_x^\alpha \rho^{(j)}(t,—)\|_{L^1_x L^1_v} = \sum_{|\alpha|\le 2d+2} \|\partial_x^\alpha \rho^{(j)}(t,—)\|_{Y^0} = \|\rho^{(j)}(t,—)\|_{\mathcal{Y}}.$$

We claim that there exists a time $T > 0$, depending only on A_0 and d, such that

$$A_j(t) \le 2A_0 \qquad \forall j \text{ and } \forall t \in [0,T]. \tag{1.2.10}$$

The proof of (1.2.10) follows from induction. Firstly, for any of the iterates,

$$A_j(0) = A_0. \tag{1.2.11}$$

Since $\rho^{(0)}$ solves the homogeneous Vlasov equation $X\rho^{(0)} = 0$ with initial condition ρ_0, conservation of mass implies

$$A_0(t) = \|\rho^{(0)}(t,—)\|_{\mathcal{Y}} = \|\rho^{(0)}(0,—)\|_{\mathcal{Y}} = A_0(0) = A_0.$$

Now assume that there exists a constant $T > 0$ depending only on the size of the initial data and d such that (1.2.10) holds for $\rho^{(j-1)}$. We will derive estimates for $\|\partial_x^\alpha \rho^{(j)}(t,—)\|_{Y^0}$ for $|\alpha| \le 2d+2$ in order to control $A_j(t) = \|\rho^{(j)}(t,—)\|_{\mathcal{Y}}$. Denote the volume form on $\mathbb{R}^{2d}_{x,v}$

as $\mathrm{d}V$ and compute with the almost conservation law (1.2.4)

$$\begin{aligned}\|\partial_x^\alpha \rho^{(j)}(t,—)\|_{Y^0} &\le \|\partial_x^\alpha \rho^{(j)}(0,—)\|_{Y^0} + \int_0^t \|\partial_x^\alpha X\rho^{(j)}(s,—)\|_{Y^0}\,\mathrm{d}s \\ &= \|\partial_x^\alpha \rho^{(j)}(0,—)\|_{Y^0} + \int_0^t \left(\iint_{\mathbb{R}^{2d}_{x,v}} \left|\partial_x^\alpha F(\rho^{(j-1)},\rho^{(j-1)})(s)\right| \right) \mathrm{d}v\mathrm{d}s \\ &\le \|\partial_x^\alpha \rho^{(j)}(0,—)\|_{Y^0} + \int_0^t \left(\iint_{\mathbb{R}^{2d}_{x,v}} \sum_{|\gamma|+|\beta|\le|\alpha|} F\left(|\partial_x^\gamma \rho^{(j-1)}|,|\partial_x^\beta \rho^{(j-1)}|\right)(s)\,\mathrm{d}v \right)\mathrm{d}s.\end{aligned} \tag{1.2.12}$$

For the sake of clarity, we examine the term in the sum above where all of the derivatives act on $\rho^{(j-1)}(t,x,v-v')$ and consider, for a fixed $v\in\mathbb{R}^d_v$, the change of variables $v+v'=w$. By Fubini-Tonelli we can interchange the order of integration

$$\begin{aligned}\iint_{\mathbb{R}^{2d}_{x,v}} F\left(|\partial_x^\alpha \rho^{(j-1)}|,|\rho^{(j-1)}|\right)(s,x,v)\,\mathrm{d}v &= \iint_{\mathbb{R}^{2d}_{x,v}}\int_{\mathbb{R}^d_w} \left|\partial_x^\alpha \rho^{(j-1)}(s,x,2v-w)\right|\left|\rho^{(j-1)}(s,x,w)\right|\,\mathrm{d}w\mathrm{d}V \\ &= \iint_{\mathbb{R}^{2d}_{x,w}} \left(\int_{\mathbb{R}^d_v} |\partial_x^\alpha \rho^{(j-1)}(s,x,2v-w)|\,\mathrm{d}v\right)|\rho^{(j-1)}(s,x,w)|\,\mathrm{d}x\mathrm{d}w.\end{aligned}$$

By Hölder's inequality this can be controlled by

$$\begin{aligned}&\le \int_{\mathbb{R}^d_x} \|\rho^{(j-1)}(s,x,\cdot)\|_{L^1(\mathbb{R}^d_w)} \left\| \int_{\mathbb{R}^d_v} |\partial_x^\alpha \rho^{(j-1)}(s,x,2v-\cdot)|\,\mathrm{d}v \right\|_{L^\infty_w} \mathrm{d}x \\ &\le \|\rho^{(j-1)}(s,—)\|_{L^\infty_x L^1_w} \int_{\mathbb{R}^d_x} \left\| \int_{\mathbb{R}^d_v} |\partial_x^\alpha \rho^{(j-1)}(s,x,2v-\cdot)|\,\mathrm{d}v \right\|_{L^\infty(\mathbb{R}^d_w)} \mathrm{d}x \\ &\le \|\rho^{(j-1)}(s,—)\|_{L^\infty_x L^1_v} \|\partial_x^\alpha \rho^{(j-1)}(s,—)\|_{L^1_x L^1_v} \\ &= \|\rho^{(j-1)}(s,—)\|_{L^\infty_x L^1_v} \|\partial_x^\alpha \rho^{(j-1)}(s,—)\|_{Y^0}.\end{aligned}$$

Note that we relabeled w and v in the first factor of the third inequality. The Sobolev inequality (1.2.5) allows us to estimate the factor without any derivatives and so the inductive hypothesis gives us

$$\begin{aligned}\iint_{\mathbb{R}^{2d}_{x,v}} F\left(|\partial_x^\alpha \rho^{(j-1)}|, |\rho^{(j-1)}|\right)(s,x,v)\, \mathrm{d}v &\leq \|\rho^{(j-1)}(s,\text{—})\|_{L^\infty_x L^1_v} \|\partial_x^\alpha \rho^{(j-1)}(s,\text{—})\|_{Y^0} \\ &\leq \|\rho^{(j-1)}(s,\text{—})\|_{\mathcal{Y}}\, \|\partial_x^\alpha \rho^{(j-1)}(s,\text{—})\|_{Y^0} \\ &\leq 2A_0 \|\partial_x^\alpha \rho^{(j-1)}(s,\text{—})\|_{Y^0}. \end{aligned} \tag{1.2.13}$$

Of course, this is only one term in the binomial expansion in (1.2.12) which comes from ∂_x^α acting on the product $\rho^{(j-1)}(s,x,v-v')\rho^{(j-1)}(s,x,v+v')$. In general, the terms will be a product

$$\partial_x^\gamma \rho^{(j-1)}(s,x,v-v')\partial_x^\beta \rho^{(j-1)}(s,x,v+v')$$

with $|\gamma|+|\beta| \leq |\alpha|$. We note that *at most* one factor in the sum will be differentiated more than $|\alpha|/2$ times. If it is the $\rho(s,x,v-v')$ term, then the same analysis that led to (1.2.13) yields

$$\begin{aligned}\iint_{\mathbb{R}^{2d}_{x,v}} F\left(|\partial_x^\gamma \rho^{(j-1)}|, |\partial_x^\beta \rho^{(j-1)}|\right)(s,x,v)\, \mathrm{d}v &\leq \|\partial_x^\beta \rho^{(j-1)}(s,\text{—})\|_{L^\infty_x L^1_v} \|\partial_x^\gamma \rho^{(j-1)}(s,\text{—})\|_{Y^0} \\ &\leq \|\rho^{(j-1)}(s,\text{—})\|_{\mathcal{Y}}\, \|\partial^\gamma \rho^{(j-1)}(s,\text{—})\|_{Y^0} \\ &\leq 2A_0 \|\partial_x^\gamma \rho^{(j-1)}(s,\text{—})\|_{Y^0} \end{aligned} \tag{1.2.13'}$$

If instead the β derivatives satisfy $|\beta| \geq |\alpha|/2$, applying the change of variables $v - v' = w$ yields the same result

$$\begin{aligned}\iint_{\mathbb{R}^{2d}_{x,v}} F\left(|\partial_x^\gamma \rho^{(j-1)}|, |\partial_x^\beta \rho^{(j-1)}|\right)(s,x,v)\, \mathrm{d}v &\leq \|\partial_x^\gamma \rho^{(j-1)}(s,\text{—})\|_{L^\infty_x L^1_v} \|\partial_x^\beta \rho^{(j-1)}(s,\text{—})\|_{Y^0} \\ &\leq \|\rho^{(j-1)}(s,\text{—})\|_{\mathcal{Y}}\, \|\partial^\beta \rho^{(j-1)}(s,\text{—})\|_{Y^0} \\ &\leq 2A_0 \|\partial_x^\beta \rho^{(j-1)}(s,\text{—})\|_{Y^0}. \end{aligned} \tag{1.2.13''}$$

We combine these results and apply them to the right hand side of (1.2.12) to find

$$\|\partial_x^\alpha \rho^{(j)}(t,—)\|_{Y^0} \le \|\partial_x^\alpha \rho^{(j)}(0,—)\|_{Y^0} + CA_0 \int_0^t \sum_{|\beta|\le|\alpha|} \|\partial_x^\beta \rho^{(j-1)}(s,—)\|_{Y^0} \, \mathrm{d}s.$$

Here C is a constant depending only on d that comes from the binomial expansion. Estimate (1.2.11), the induction hypothesis, and summing up over over the multi-indices of length less than or equal to $2d+2$ yields

$$\begin{aligned} A_j(t) &\le A_j(0) + CA_0 \int_0^t A_{j-1}(s) \, \mathrm{d}s \\ &\le A_0 + CA_0 \int_0^t 2A_0 \, \mathrm{d}s \\ &= A_0 + CA_0^2 t. \end{aligned}$$

Choose $T = 1/(CA_0)$ to find that $A_j(t) \le 2A_0$ for all $t \in [0,T]$. This concludes the induction and so (1.2.10) holds for all j.

Let $T = 1/CA_0$ be the time constructed for estimate (1.2.10) for the initial data $\rho_0 \in Y$ with norm $\|\rho_0\|_{\mathcal{Y}} = A_0$. We now show that equation (1.2.1) has a unique solution on $[0,T']$ for some $T' < T$ to be determined later. Denote the balls

$$B(R) = \{\rho \in C^0([0,T'];\mathcal{Y}) \mid \|\rho\|_{C^0([0,T'];\mathcal{Y})} < R\} \qquad \text{and} \qquad B^{\mathcal{Y}}(R) = \{\rho \in \mathcal{Y} \mid \|\rho\|_{\mathcal{Y}} < R\}$$

for any $R > 0$. Define the map G_{ρ_0} by

$$G_{\rho_0} : \rho(t,x,v) \mapsto \rho_0(x - tv, v) + \int_0^t F(\rho,\rho)(s, x + (s-t)v, v) \, \mathrm{d}s. \tag{1.2.14}$$

One explicitly checks that

$$\begin{cases} X(G_{\rho_0}(\rho)) = F(\rho,\rho) \\ G_{\rho_0}(\rho)(0,x,v) = \rho_0(x,v) \end{cases}$$

and so a solution to equation (1.2.1) would be given by a fixed point $G_{\rho_0}(\rho) = \rho$. Estimate (1.2.10) and a density argument show that G_{ρ_0} is actually a map $G_{\rho_0} : B(2A_0) \to B(2A_0)$.

Banach's Fixed Point Theorem will show existence of a unique ρ such that $G_{\rho_0}(\rho) = \rho$ provided that G_{ρ_0} is a contraction mapping. Let $\rho, \rho' \in B(2A_0)$ be arbitrary functions. Existence and uniqueness of (1.2.1) will then follow from

$$\|G_{\rho_0}(\rho) - G_{\rho_0}(\rho')\|_{C^0([0,T'];\mathcal{Y})} \le \gamma \|\rho - \rho'\|_{C^0([0,T'];\mathcal{Y})} \tag{1.2.15}$$

for some $0 < \gamma < 1$. We will prove (1.2.15) and Lipschitz dependence on initial data simultaneously.

Pick two arbitrary $\rho_0, \rho_0' \in B^{\mathcal{Y}}(A_0)$. The function $G_{\rho_0}(\rho) - G_{\rho_0'}(\rho')$ solves

$$\begin{cases} X(G_{\rho_0}(\rho) - G_{\rho_0'}(\rho')) = F(\rho,\rho) - F(\rho',\rho') \\ (G_{\rho_0'}(\rho) - G_{\rho_0'}(\rho'))(0,x,v) = \rho_0(x,v) - \rho_0'(x,v). \end{cases}$$

The conservation law (1.2.4) shows

$$\|(G_{\rho_0}(\rho) - G_{\rho_0'}(\rho'))(t,—)\|_Y \le \|\rho_0 - \rho_0'\|_Y + \int_0^t \|(F_i(\rho) - F_i(\rho'))(s,—)\|_Y \, \mathrm{d}s. \tag{1.2.16}$$

For a fixed multi-index α the $L^1_x L^1_v$ $(= Y^0)$ norm of the difference $\partial_x^\alpha F(\rho,\rho) - \partial_x^\alpha F(\rho',\rho')$ is equivalent to

$$\begin{aligned} &\iint_{\mathbb{R}^{2d}_{x,v}} \left|\partial_x^\alpha (F(\rho,\rho) - F(\rho',\rho'))(t)\right| \, \mathrm{d}v \\ &\qquad = \iint_{\mathbb{R}^{2d}_{x,v}} \left| \sum_{|\gamma|+|\beta|\le|\alpha|} F\left(\partial_x^\gamma \rho, \partial_x^\beta \rho - \partial_x^\beta \rho'\right)(t) + F\left(\partial_x^\gamma \rho - \partial_x^\gamma \rho', \partial_x^\beta \rho'\right)(t) \right| \mathrm{d}V. \end{aligned}$$

With this we reduce our attention to the two integrals

$$\begin{aligned} I_1(t) &\overset{\text{def}}{=} \iint_{\mathbb{R}^{2d}_{x,v}} \sum_{|\gamma|+|\beta|\le|\alpha|} F\left(|\partial_x^\gamma \rho|, |\partial_x^\beta \rho - \partial_x^\beta \rho'|\right)(t) \, \mathrm{d}V \\ I_2(t) &\overset{\text{def}}{=} \iint_{\mathbb{R}^{2d}_{x,v}} \sum_{|\gamma|+|\beta|\le|\alpha|} F\left(|\partial^\gamma \rho - \partial_x^\gamma \rho|, |\partial_x^\beta \rho'|\right)(t) \, \mathrm{d}V. \end{aligned}$$

We note that when $|\gamma| \geq |\alpha|/2$, the Sobolev inequality bounds the $L^\infty_x L^1_v$ norm of the difference terms $\partial^\beta_x \rho - \partial^\beta_x \rho'$ in $I_1(t)$ by their Y^0 norms and the $\partial^\beta_x \rho'$ terms in $I_2(t)$ by their $\mathcal{Y}$ norms, and vice-versa when $|\beta| \geq |\alpha|/2$. Keeping the difference terms $\rho - \rho'$ in the integral norms and bounding the monomial terms of ρ, ρ' by their $\mathcal{Y}$ norms, and using $G : B(2A_0) \to B(2A_0)$ shows

$$\begin{aligned} I_1(t) + I_2(t) &\leq C \sum_{|\beta| \leq |\alpha|} \|(\partial^\beta_x \rho - \partial^\beta_x \rho')(t, —)\|_{L^1_x L^1_v} (2A_0 + 2A_0) \\ &= 4CA_0 \sum_{|\beta| \leq |\alpha|} \|(\partial^\beta_x \rho - \partial^\beta_x \rho')(t, —)\|_{Y^0}. \end{aligned}$$

Here C is a constant depending only on d that comes from the binomial expansion. We apply this estimate to (1.2.16) to find

$$\begin{aligned} \|(G_{\rho_0}(\rho) - G_{\rho'_0}(\rho'))(t, —)\|_Y &\leq \|\rho_0 - \rho'_0\|_Y + \int_0^t \|(F(\rho, \rho) - F(\rho', \rho'))(s, —)\|_Y \, \mathrm{d}s \\ &\leq \|\rho_0 - \rho'_0\|_Y + \int_0^t 4CA_0 \sum_{\substack{|\beta| \leq |\alpha|, \\ |\alpha| \leq 2d+2}} \|(\partial^\beta_x \rho - \partial^\beta_x \rho')(s, —)\|_{Y^0} \, \mathrm{d}s \\ &\leq \|\rho_0 - \rho'_0\|_Y + \int_0^t 4CA_0 \|(\rho - \rho')(s, —)\|_{\mathcal{Y}} \, \mathrm{d}s. \end{aligned} \tag{1.2.17}$$

Note that if $\rho_0 = \rho'_0$ the first term in inequality (1.2.17) vanishes. Then

$$\begin{aligned} \|(G_{\rho_0}(\rho) - G_{\rho_0}(\rho'))(t, —)\|_{\mathcal{Y}} &\leq 4CA_0 \int_0^t \|(\rho - \rho')(s, —)\|_{\mathcal{Y}} \, \mathrm{d}s \\ &\leq 4CA_0 \|\rho - \rho'\|_{C^0([0,T'];\mathcal{Y})} \int_0^t 1 \, \mathrm{d}s \\ &\leq 4CA_0 \|\rho - \rho'\|_{C^0([0,T'];\mathcal{Y})} t. \end{aligned}$$

Choose the time of existence to be controlled by $T' < \frac{1}{8CA_0}$. Then the right hand side is bounded by $\frac{1}{2}\|\rho - \rho'\|_{C^0([0,T];\mathcal{Y})}$. This does not depend on $t \in [0, T']$ so we can take the supremum to find

$$\|G_{\rho_0}(\rho) - G_{\rho_0}(\rho')\|_{C^0([0,T'];Y)} \le \frac{1}{2}\|\rho - \rho'\|_{C^0([0,T];\mathcal{Y})}.$$

This concludes the proof of (1.2.15) and so the fixed point $G_{\rho_0}(\rho) = \rho$ is the unique solution of (1.2.1).

Now suppose that $\rho_0 \neq \rho_0'$ with $\rho_0, \rho_0' \in B^{\mathcal{Y}}(A_0)$. Let ρ, ρ' be the two solutions of (1.2.1) with these respective initial conditions. Applying Gronwall's inequality to (1.2.17) shows

$$\begin{aligned}
\|(\rho - \rho')(s,\cdot)\|_{\mathcal{Y}} &\le \|\rho - \rho_0'\|_{\mathcal{Y}} \exp\left(\int_0^t 4CA_0 \, \mathrm{d}s\right) \\
&= \|\rho - \rho_0'\|_{\mathcal{Y}} \exp(4CA_0 t) \\
&\le \|\rho - \rho_0'\|_{\mathcal{Y}} \exp\left(\tfrac{4CA_0}{8CA_0}\right) \\
&\le C\|\rho - \rho_0'\|_{\mathcal{Y}}.
\end{aligned}$$

Taking the supremum over $[0,T']$ shows

$$\|\rho - \rho'\|_{C^0([0,T'];\mathcal{Y})} \le C\|\rho_0 - \rho_0'\|_{\mathcal{Y}}.$$

This concludes the proof of Lipschitz dependence of initial data and the theorem. □

Remark 1.2.6. We have shown that, given a $\rho_0 \in Y^{2d+2}$, there exists a $T > 0$ depending on the size of ρ_0 such that (1.2.1) has unique solution $\rho \in C^0([0,T];Y^{2d+2})$ with ρ_0 as its initial value. That is, ρ satisfies

$$\partial_t\rho + v \cdot \partial_x\rho = F(\rho,\rho).$$

The function ρ is actually in $C^0([0,T];Y^{2d+2}) \cap C^1([0,T];Y^{2d+1})$ because

$$\partial_t\rho = -v \cdot \partial_x\rho + F(\rho,\rho)$$

and the right hand side is in $C^0([0,T];Y^{2d+1})$.

With local wellposedness established, we now introduce a key technical tool used for proving global existence: the breakdown criterion.

Theorem 1.2.7 (Breakdown criterion). *Define*

$$T_* := \sup\{T > 0 \mid \exists \rho \in C^0([0,T]; Y^{2d+2}) \text{ solving (1.2.1)}\}. \tag{1.2.18}$$

Then either $T_* = \infty$ or

$$\sum_{|\alpha| \le d+1} \|\partial_x^\alpha \rho(t,x,\cdot)\|_{L^1_v(\mathbb{R}^d)} \notin L^\infty([0,T_*) \times \mathbb{R}^d_x). \tag{1.2.19}$$

Proof. The proof follows by a contradiction argument. If $T_* = \infty$ then there is nothing to prove so suppose that $T_* < \infty$. Assume that (1.2.19) does not hold. Then we claim that

$$A_0 \stackrel{\text{def}}{=} \sup_{t \in [0,T_*)} \sum_{|\alpha| \le d+1} \|\partial_x^\alpha \rho(t, —)\|_{L^\infty_x L^1_v} < \infty \tag{1.2.20}$$

implies

$$A(t) \stackrel{\text{def}}{=} \|\rho(t, —)\|_{Y^{2d+2}} = \sum_{|\alpha| \le 2d+2} \|\partial_x^\alpha \rho(t, —)\|_{L^1_x L^1_v} < \infty \tag{1.2.21}$$

for all $t \in [0, T_*)$. Fix a multi-index α with length $|\alpha| \le 2d+2$. The conservation of mass estimate (1.2.8) gives

$$\|\partial_x^\alpha \rho(t, —)\|_{Y^0} \le \|\partial_x^\alpha \rho(0, —)\|_{Y^0} + \int_0^t \|\partial_x^\alpha X \rho(s, —)\|_{Y^0} \, \mathrm{d}s. \tag{1.2.22}$$

The term in the integral is explicitly

$$\iint_{\mathbb{R}^{2d}_{x,v}} \left|\partial_x^\alpha F(\rho,\rho)(s,x,v)\right| \mathrm{d}v \le \iint_{\mathbb{R}^{2d}_{x,v}} \sum_{|\gamma|+|\beta| \le |\alpha|} F\left(|\partial_x^\gamma \rho|, |\partial_x^\beta \rho|\right) \mathrm{d}V.$$

Then either $|\gamma| \ge |\alpha|/2$ or $|\beta| \ge |\alpha|/2$ in each term of the sum. In the former case, the change of variables $v + v' = w$ and the same analysis as in (1.2.13') shows

$$\iint_{\mathbb{R}^{2d}_{x,v}} \int_{\mathbb{R}^d_{v'}} |\partial_x^\gamma \rho(s,x,v-v')| |\partial_x^\beta \rho(s,x,v+v')| \, \mathrm{d}v' \mathrm{d}V \le \|\partial_x^\gamma \rho(s, —)\|_{L^1_x L^1_v} \|\partial_x^\beta \rho(s, —)\|_{L^\infty_x L^1_v}.$$

In the latter case the change of variables $v - v' = w$ yields

$$\iint_{\mathbb{R}^{2d}_{x,v}} \int_{\mathbb{R}^d_{v'}} |\partial_x^\gamma \rho(s,x,v-v')| |\partial_x^\beta \rho(s,x,v+v')| \, \mathrm{d}v' dV \le \|\partial_x^\beta \rho(s, —)\|_{L^1_x L^1_v} \|\partial_x^\gamma \rho(s, —)\|_{L^\infty_x L^1_v}.$$

In both instances, the factor in the L^∞ norm will be differentiated at most $d+1$ times because $|\alpha| \leq 2d+2$. The assumption (1.2.20) allows us to bound these factors by A_0. Applying this estimate to (1.2.22), summing over all multi-indices, and using Gronwall's inequality yields

$$\sum_{|\alpha|\leq 2d+2} \|\partial_x^\alpha \rho(t,—)\|_{Y^0} \leq \sum_{|\alpha|\leq 2d+2} \|\partial_x^\alpha \rho(0,—)\|_{Y^0} + A_0 \int_0^t \sum_{|\beta|\leq|\alpha|} \|\partial_x^\beta \rho(s,—)\|_{Y^0}\, \mathrm{d}s$$

$$\begin{aligned} A(t) &\leq A(0) + A_0 \int_0^t A(s)\, \mathrm{d}s \\ A(t) &\leq A(0) \exp\left(\int_0^t A_0\, \mathrm{d}s \right) \\ &= A(0)\exp(tA_0) \leq A(0) e^{T_* A_0} \\ &< \infty. \end{aligned}$$

The last inequality follows from the assumption $T_* < \infty$ and the initial data assumption $\rho_0 \in Y^{2d+2}$. This concludes the proof of the claim (1.2.21). Now let $\{t_n\}$ be an increasing sequence such that $t_n \nearrow T_*$ as $n \to \infty$. The claim shows that $\rho(t_n,—)$ can be taken as an initial condition for the Cauchy problem and so Theorem 1.2.2 shows that there exists a small $T > 0$ such that a solution exists on $[t_n, t_n + T]$. The uniqueness of the solution shows that ρ can be extended past T_*, contradicting the definition of supremum. □

1.2.6 Vector field method in action

It remains to show global existence for equation (1.2.1). In order to exploit the breakdown criterion, we need to show

$$\sup_{t\in[0,T_*)} \sum_{|\alpha|\leq d+1} \|\partial_x^\alpha \rho(t,—)\|_{L_x^\infty L_v^1} < \infty, \tag{1.2.23}$$

where $\rho : [0, T_*) \times \mathbb{R}^2_{x,v} \to \overline{\mathbb{R}_+}$ is a solution to (1.2.1) and T_* is as in (1.2.18). The Sobolev inequality states

$$\sum_{|\alpha| \le d+1} \|\partial_x^\alpha \rho(t, —)\|_{L_x^\infty L_v^1} \le C_d \|\rho(t, —)\|_{Y^{2d+1}}$$

for all $t \in [0, T_*)$. This shows that getting a bound

$$\|\rho(t, —)\|_{Y^{2d+1}} \le C \tag{1.2.24}$$

which is uniform in time will prove (1.2.23) and, as a consequence of the breakdown criterion of Theorem 1.2.7, show that $T_* = \infty$.

This approach of constructing a global solution is akin to the proof of local wellposedness in Theorem 1.2.2. Estimate (1.2.24) which is uniform in time is analogous to the uniform estimate (1.2.10); the proof of which was an induction argument. In the global existence scenario the induction argument manifests itself as a bootstrapping argument where we will show that the desired bound holds on a non-empty open and closed subset of $[0, T_*)$ (and hence on all of $[0, T_*)$ itself). Appendix A.1 shows that, in fact, the bootstrap mechanism can be thought of as a continuous induction argument.

A fundamental tool in closing the induction for estimate (1.2.10) was the the commutation relation $[X, \partial_x^\alpha] = X\partial_x^\alpha - \partial_x^\alpha X = 0$. This identity immediately revealed the Vlasov-type equation solved by the derivatives $\partial_x^\alpha \rho$, to which the approximate-conservation of mass result (1.2.4) can apply. Indeed, differentiating the equation that the Picard iterates solved produced an equation for their derivatives:

$$X\partial_x^\alpha \rho^{(j)} = \partial_x^\alpha X\rho^{(j)} = \partial_x^\alpha F(\rho^{(j-1)}, \rho^{(j-1)}) = \sum_{|\gamma|+|\beta|=|\alpha|} F\left(\partial_x^\gamma \rho^{(j-1)}, \partial_x^\beta \rho^{(j-1)}\right).$$

Careful analysis of the right hand side involving the conservation estimate (1.2.8), the Sobolev inequality (A.1.2), and the induction hypothesis led to (1.2.10).

The vector field method is based on finding a collection of temporally *weighted* vector fields $\{W\}$ that preserve the linear part of (1.2.1); they too satisfy the commutation relation $[X, W^\alpha] = XW^\alpha - W^\alpha X = 0$. *In direct analogy with the LWP scenario*, this produces an equation for the W-derivatives of ρ:

$$XW^{\alpha}\rho = W^{\alpha}X\rho = W^{\alpha}F(\rho,\rho) = \sum_{|\gamma|+|\beta|=|\alpha|} F\left(W^{\gamma}\rho, W^{\beta}\rho\right).$$

This corresponds to Point **I** of the general strategy for the vector field method introduced in the Prologue of this chapter.

A second key step in the vector field method is to use the temporal weight of $\{W\}$ and the L^2—L^∞ Sobolev inequality (A.1.2) to obtain an estimate that reflects the dispersive decay that the density satisfies, see Proposition 1.2.10 and Remark 1.2.8. Our approach for proving this estimate is adapted from Smulevici et al [Smu16, FJS17]. To control the terms $F(W^{\gamma}\rho, W^{\beta}\rho)$ that appear in the conservation law, the decay estimate is supplemented with the aforementioned bootstrap assumption, where we *assume* that the estimate (1.2.24) holds. To avoid circularity, we use the temporal decay to show that the right hand side of estimate (1.2.24) can be replaced with $C/2$. *This is in direct analogy with the LWP scenario*, where the inductive hypothesis of A_{j-1} was used to show that it also holds for A_j. This analysis is a nonlinear modification of Points **II** and **III** introduced in the Prologue.

Remark 1.2.8. In fact, the pointwise decay in time will be proven for the *spatial density* $\overline{\rho}(t,x) = \int_{\mathbb{R}^d_v} \rho(t,x,v)\, \mathrm{d}v$. That this is available for the integral over velocity arises from the intuition that dispersion occurs because the physical extent of the particles spreads out while the total mass is conserved. With the same mass divided among a greater volume, the spatial density (which is what $\overline{\rho}$ measures) must decay.

1.2.6.1 The weighted vectorfields

Galilean relativity is the physical theory that asserts that the the laws of motion prescribed by Newtonian mechanics are the same in all inertial frames. Newtonian interaction of particles reduces to, under the simplification that individual particles don't interact, Newton's first law of motion: the Vlasov equation

$$\partial_t \rho(t,x,v) + v \cdot \partial_x \rho(t,x,v) = X\rho = 0.$$

The *Galilean boost* transformation of $\rho(t,x,v)$ is, for any $w \in \mathbb{R}^d$, the change of inertial frame defined by

$$\rho_w(t,x,v) := \rho(t, x + tw, v + w). \tag{1.2.25}$$

This action is continuously parametrized by w and acts as the identity when $w = 0$. An explicit computation shows that if ρ solves the linear Vlasov equation then ρ_w does indeed solve it as well:

$$X(\rho_w) = 0.$$

Assuming the family ρ_w is differentiable in w, the linearity of the Vlasov operator X implies that $\frac{\mathrm{d}\rho_w}{dw}$ at any $w = w'$ also solves the Vlasov equation:

$$X\left(\frac{\mathrm{d}\rho_w}{dw}\bigg|_{w=w'} \right) = 0.$$

Taking $w = e_i$ as the basis element in $\mathbb{R}^d$ we find $\frac{\mathrm{d}\rho_w}{dw}\Big|_{w=e_i} = t\partial_{x^i}\rho + \partial_{v^i}\rho$. We hence define

$$W_i := t\partial_{x^i} + \partial_{v^i} \tag{1.2.26}$$

for $i = 1, \dots, d$, which are the infinitesimal generators of the Galilean boosts (1.2.25). This construction of the W_i shows that they satisfy the commutation relation

$$[X, W_i] = 0, \quad \forall i = 1, \dots, d. \tag{1.2.27}$$

Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be any multi-index and denote $W^\alpha = W_1^{\alpha_1} \cdots W_d^{\alpha_d}$. Then (1.2.27) and an induction argument show $[X, W^\alpha] = 0$ for all multi-indices. These computations, and the t-weight in each W_i, imply that they are good candidates for getting pointwise time decay for the Vlasov equation. Denote

$$\overline{\rho}(t,x) := \int_{\mathbb{R}^d_v} \rho(t,x,v)\,\mathrm{d}v \tag{1.2.28}$$

as the spatial density of ρ. The following lemma is the first step in proving our desired result.

Lemma 1.2.9. *Let* $\rho \in C_c^\infty(\mathbb{R}_t \times^{2d}_{x,v})$ *and suppose* $t > 0$. *Then*

$$|\overline{\rho}(t,x)| \leq \frac{1}{t^d} \sum_{|\alpha| \leq d} \int_{\mathbb{R}^{2d}_{y,v}} |W^\alpha \rho(t,y,v)| \, \mathrm{d}y \mathrm{d}v. \tag{1.2.29}$$

Proof. For $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$, define the infinite rectangle

$$R_x := \{y \in \mathbb{R}^d \mid y^i \leq x^i \; \forall i = 1, \ldots d\}.$$

The compact support of ρ implies

$$\begin{aligned}\overline{\rho}(t,x) &= \int_{R_x} \partial_{x^1} \cdots \partial_{x^d} \overline{\rho}(t,y) \, \mathrm{d}y \\ &= \int_{R_x} \int_{\mathbb{R}^d_v} \partial_{x^1} \cdots \partial_{x^d} \rho(t,y,v) \, \mathrm{d}y \mathrm{d}v.\end{aligned}$$

The compact support also tells us

$$\int_{\mathbb{R}^d_v} \partial_{v^i} \rho(t,x,v) \, \mathrm{d}v = 0$$

for all $i = 1, \ldots, d$. We then arrive to

$$\begin{aligned}|\overline{\rho}(t,x)| &= \left| \int_{R_x} \int_{\mathbb{R}^d_v} \partial_{x^1} \cdots \partial_{x^d} \rho(t,y,v) \, dy dv \right| = \frac{1}{t^d} \left| \int_{R_x} \int_{\mathbb{R}^d_v} W_1 \cdots W_d \rho(t,y,v) \, dy dv \right| \\ &\leq \frac{1}{t^d} \int_{\mathbb{R}^{2d}_{y,v}} |W_1 \cdots W_d \rho(t,y,v)| \, dy dv \leq \frac{1}{t^d} \sum_{|\alpha| \leq d} \int_{\mathbb{R}^{2d}_{y,v}} |W^\alpha \rho(t,y,v)| \, dy dv.\end{aligned}$$

This is the desired inequality so the proof is complete. □

We note that the proof of Lemma 1.2.9 provided a sharper estimate than (1.2.29); we had bounded $|\rho(t,x)|$ with the same decay but each W_i was used only once. We also note that the estimate is not useful for $t = 0$ because the right hand side blows up as $t \searrow 0$. We now show how to overcome this hurdle. Let $0 \le t \le 1$ and define the Japanese bracket $\langle t \rangle = \sqrt{1+t^2}$. Then $\langle t \rangle \le \sqrt{2}$ so this immediately implies that $\langle t \rangle^{-d} \ge 2^{-d/2}$. Then

$$\begin{aligned}
|\overline{\rho}(t,x)| &= \left| \int_{R_x} \int_{\mathbb{R}^d_v} \partial_{x^1} \cdots \partial_{x^d} \rho(t,y,v)\, dy dv \right| \le \int_{\mathbb{R}^{2d}_{y,v}} |\partial_{x^1} \cdots \partial_{x^d} \rho(t,y,v)|\, dy dv \\
&\le \sum_{|\alpha| \le d} \int_{\mathbb{R}^{2d}_{y,v}} |\partial_x^\alpha \rho(t,y,v)|\, dy dv = \frac{2^{d/2}}{2^{d/2}} \sum_{|\alpha| \le d} \int_{\mathbb{R}^{2d}_{y,v}} |\partial_x^\alpha \rho(t,y,v)|\, dy dv \\
&\le \frac{C}{\langle t \rangle^d} \sum_{|\alpha| \le d} \int_{\mathbb{R}^{2d}_{y,v}} |\partial_x^\alpha \rho(t,y,v)|\, dy dv. \qquad (1.2.30)
\end{aligned}$$

This estimate fixes the weakness of Lemma 1.2.9. Consider the collection $\{\partial_{x^i}, W_i\}$ and denote Γ as an arbitrary element of this collection. Define Γ^α in the natural way. Then we have proved the following.

Proposition 1.2.10 (Pointwise decay for Vlasov)**.** *Let $\rho \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}^{2d}_{x,v})$ and $t \ge 0$. Then there exists a constant C depending only on d such that*

$$|\overline{\rho}(t,x)| \le \frac{C}{\langle t \rangle^d} \sum_{|\alpha| \le d} \int_{\mathbb{R}^{2d}_{y,v}} |\Gamma^\alpha \rho(t,y,v)|\, dy dv. \qquad (1.2.31)$$

Remark 1.2.11. We note that the proof for (1.2.31) does not need the full power of $\rho \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}^{2d}_{x,v})$. Our theorem will have $\rho \in C^0([0,T]; Y^s) \cap C^1([0,T]; Y^{s-1})$ for some s so taking spatial derivatives, at least in the weak sense, is valid so long as s is large enough. However, we do not have any regularity in v from our LWP theorem. The upshot is that we do not need it; we only need that ρ is well defined on integral curves of W_i so that the derivatives $W_i \rho$ makes sense.

1.2.7 Global existence

Proof of Theorem 1.2.3. We present the proof of (a) first. Assume the spatial dimension satisfies $d \geq 2$. Theorems 1.2.2 and 1.2.7 show that $T_*^\epsilon > 0$ for every $\epsilon > 0$. We need to show that $T_*^\epsilon = \infty$ for small enough initial data. Assume that $T_*^\epsilon < \infty$ for all $\epsilon > 0$. Then Theorem 1.2.7 implies

$$\sum_{|\alpha| \leq d+1} \|\partial_x^\alpha \rho(t,x,\cdot)\|_{L_v^1(\mathbb{R}^d)} \notin L^\infty([0,T_*^\epsilon) \times \mathbb{R}_x^d).$$

We will arrive to a contradiction by showing that there exists a small $\epsilon_0 > 0$ such that

$$\sup_{(t,x) \in [0,T_*^\epsilon) \times \mathbb{R}_x^d} \sum_{|\alpha| \leq d+1} \|\partial_x^\alpha \rho(t,x,\cdot)\|_{L_v^1(\mathbb{R}^d)} < \infty \tag{1.2.32}$$

for all $\epsilon < \epsilon_0$. Define the weighted energy as

$$A(t) := \sum_{|\alpha| \leq 2d+1} \|\Gamma^\alpha \rho(t, —)\|_{L_x^1 L_v^1}.$$

We will prove (1.2.32) by showing that

$$A(t) \leq A\epsilon, \qquad 0 \leq t \leq T_*^\epsilon \tag{1.2.33}$$

for small enough $\epsilon > 0$ and some $A > 0$ that is independent of time. First we show why (1.2.33) implies (1.2.32). Fix a multi-index α of length $|\alpha| \leq d+1$. Then the pointwise estimate (1.2.31) shows

$$\begin{aligned}
\sum_{|\alpha| \leq d+1} \|\partial_x^\alpha \rho(t,x,\cdot)\|_{L_v^1(\mathbb{R}^d)} &\leq \sum_{|\alpha| \leq d+1} \|\Gamma^\alpha \rho(t,x,\cdot)\|_{L_v^1(\mathbb{R}^d)} \\
&\leq \frac{C}{\langle t \rangle^d} \sum_{|\beta| \leq d} \sum_{|\alpha| \leq d+1} \|\Gamma^\beta \Gamma^\alpha \rho(t, —)\|_{L_x^1 L_v^1} \\
&\leq \frac{C}{\langle t \rangle^d} A(t) \leq \frac{CA\epsilon}{\langle t \rangle^d} < \infty.
\end{aligned} \tag{1.2.34}$$

Taking the supremum over all $(t,x) \in [0,T_*^\epsilon) \times \mathbb{R}_x^d$ proves (1.2.32) and the theorem. Hence, it suffices to show that there exists an $\epsilon_0 > 0$ such that (1.2.33) holds for all $\epsilon < \epsilon_0$. This is done by bootstrapping.

We note that the initial energy $A(0)$ depends only on the initial data. Hence we can find a large constant A depending on $2d+1$ derivatives of ρ_0 (but is independent of ϵ) such that $A(0) \leq A\epsilon$. Define the set

$$E_\epsilon := \{T \in [0, T_*^\epsilon) \mid A(t) \leq A\epsilon \ \forall 0 \leq t \leq T\}. \tag{1.2.35}$$

This set is not empty because $0 \in E_\epsilon$ by construction. Moreover, $A(t) \in C([0, T_*^\epsilon))$ so E_ϵ is also closed. By the continuity principle, (1.2.33) will be proved if we show that E_ϵ is open because it would imply $E_\epsilon = [0, T_*^\epsilon)$.

Let $t_0 \in E_\epsilon$. Continuity of $A(t)$ shows that there exists a $T' > t_0$ such that

$$|A(T') - A(t_0)| \leq A\epsilon,$$

and so this immediately implies

$$A(t) \leq 2A\epsilon, \qquad 0 \leq t \leq T'. \tag{1.2.36}$$

We show that $A(t) \leq 2A\epsilon$ actually implies the sharper bound $A(t) \leq A\epsilon$ for small enough ϵ. This would imply that E_ϵ is open, as we hoped. Let α be a fixed multi-index of length $|\alpha| \leq 2d+1$. The commutation relation $X\Gamma^\alpha = \Gamma^\alpha X$ and the proof for (1.2.8) immediately imply

$$\|\Gamma^\alpha \rho(t, —)\|_{Y^0} \leq \|\Gamma^\alpha \rho(0, —)\|_{Y^0} + \int_0^t \|\Gamma^\alpha X \rho(s, —)\|_{Y^0}. \tag{1.2.37}$$

The integral term is precisely

$$\iint_{\mathbb{R}^{2d}_{x,v}} \left|\Gamma^\alpha F(\rho,\rho)(s,x,v)\right| \, \mathrm{d}v = \iint_{\mathbb{R}^{2d}_{x,v}} \left| \sum_{|\gamma|+|\beta| \leq |\alpha|} F\left(\Gamma^\gamma \rho, \Gamma^\beta \rho\right)(s,x,v) \right| \, \mathrm{d}v.$$

Then either $|\gamma| \geq |\alpha|/2$ or $|\beta| \geq |\alpha|/2$. In the former case, we do the change of variables $v + v' = w$, use the same analysis as in (1.2.13), apply the pointwise decay estimate (1.2.31), and the bootstrapping assumption to find

$$\begin{aligned}\iint\limits_{\mathbb{R}^{2d}_{x,v}} F\Big(|\Gamma^{\gamma}\rho|, |\Gamma^{\beta}\rho|\Big)(s,x,v)\ \mathrm{d}v &\le \|\Gamma^{\gamma}\rho(s,—)\|_{L^1_xL^1_v}\|\Gamma^{\beta}\rho(s,—)\|_{L^\infty_xL^1_v}\\ &\le \frac{C}{\langle s\rangle^d}\|\Gamma^{\gamma}\rho(s,—)\|_{L^1_xL^1_v}\sum_{|\delta|\le d+|\beta|}\|\Gamma^{\delta}\rho(s,—)\|_{L^1_xL^1_v}\\ &\le \frac{2CA\epsilon}{\langle s\rangle^d}\|\Gamma^{\gamma}\rho(s,—)\|_{Y0}.\end{aligned}$$

In the latter case the change of variables $v - v' = w$ and a similar argument yields

$$\begin{aligned}\iint\limits_{\mathbb{R}^{2d}_{x,v}} F\Big(|\Gamma^{\gamma}\rho|, |\Gamma^{\beta}\rho|\Big)(s,x,v)\ \mathrm{d}v &\le \|\Gamma^{\beta}\rho(s,—)\|_{L^1_xL^1_v}\|\Gamma^{\gamma}\rho(s,—)\|_{L^\infty_xL^1_v}\\ &\le \frac{C}{\langle s\rangle^d}\|\Gamma^{\beta}\rho(s,—)\|_{L^1_xL^1_v}\sum_{|\delta|\le d+|\gamma|}\|\Gamma^{\delta}\rho(s,—)\|_{L^1_xL^1_v}\\ &\le \frac{2CA\epsilon}{\langle s\rangle^d}\|\Gamma^{\beta}\rho(s,—)\|_{Y0}.\end{aligned}$$

These estimates show that

$$\|\Gamma^{\alpha}X\rho(s,—)\|_{Y0} \le \sum_{|\beta|\le|\alpha|}\frac{2CA\epsilon}{\langle s\rangle^d}\|\Gamma^{\beta}\rho(s,—)\|_{Y0}$$

for all fixed indeces. We put this bound into (1.2.37) and sum over all multiindices α to find

$$\begin{aligned}\sum_{|\alpha|\le 2d+1}\|\Gamma^{\alpha}\rho(t,—)\|_{Y0} &\le \sum_{|\alpha|\le 2d+1}\|\Gamma^{\alpha}\rho(0,—)\|_{Y0} + \sum_{|\alpha|\le 2d+1}\int_0^t \|\Gamma^{\alpha}X\rho(s,—)\|_{Y0}\ \mathrm{d}s\\ &\le A(0) + \sum_{|\alpha|\le 2d+1}\sum_{|\beta|\le|\alpha|}\int_0^t \frac{2CA\epsilon}{\langle s\rangle^d}\|\Gamma^{\beta}\rho(s,—)\|_{L^1_xL^1_v}\ \mathrm{d}s\\ &\le A(0) + \int_0^t \frac{2CA\epsilon}{\langle s\rangle^d}A(s)\ \mathrm{d}s.\end{aligned}$$

We can finally apply Gronwall's inequality to see

$$\begin{aligned} A(t) &\leq A(0)\exp\left(\int_0^t \frac{2CA\epsilon}{\langle s\rangle^d}\,\mathrm{d}s\right) \\ &\leq A\epsilon\exp\left(\int_0^\infty \frac{2CA\epsilon}{\langle s\rangle^d}\,\mathrm{d}s\right). \end{aligned} \tag{1.2.38}$$

Choose $\epsilon_0 > 0$ small enough such that the exponential term is bounded by 1, which can be done because $\langle s\rangle^{-d}$ is integrable for $d \geq 2$. Then (1.2.38) implies $A(t) \leq A\epsilon$ for all $\epsilon < \epsilon_0$ and the proof of (a) is complete.

We now focus our attention on (b) of Theorem 1.2.3. Assume that $d = 1$ and that ρ is a solution to (1.1_ϵ) on $[0, T]$. The proof of Theorem 1.2.7 shows that T can be extended by a positive value so long as

$$\sup_{t\in[0,T)} \sum_{|\alpha|\leq d+1} \|\partial_x^\alpha \rho(t,—)\|_{L_x^\infty L_v^1(\mathbb{R}^{2d})} < \infty.$$

We prove the desired lower bound for T_*^ϵ by showing that T can be extended until

$$T^\epsilon = \sinh\left(\frac{B}{\epsilon}\right)$$

for some B and ϵ to be determined later. As in the proof of (1.2.34), this will be accomplished by showing that

$$A(t) \leq A\epsilon, \qquad \forall 0 \leq t \leq T^\epsilon \tag{1.2.39}$$

where A (which depends on three derivatives of the initial data, but is independent of ϵ) is as in $A(0) \leq A\epsilon$. Define the set

$$E_\epsilon := \{T \in [0, T^\epsilon) \mid A(t) \leq A\epsilon\ \forall 0 \leq t \leq T\}.$$

This set is not empty because $0 \in E_\epsilon$ by construction. Moreover, $A(t) \in C([0, T^\epsilon))$ so E_ϵ is also closed. Let $t_0 \in E_\epsilon$. Continuity of $A(t)$ shows that there exists a $T' > t_0$ such that

$$A(t) \leq 2A\epsilon, \qquad 0 \leq t \leq T'. \tag{1.2.40}$$

This is the bootstrapping assumption. As in the proof of (a), we show that (1.2.40) implies $A(t) \leq A\epsilon$ for $0 \leq t \leq T'$. This would imply the openness of E_ϵ and consequently $E_\epsilon = [0, T^\epsilon)$ by the continuity principle.

Choose $B > 0$ small enough so that $\exp(2CAB) < 1$ and $\epsilon_0 > 0$ small enough so that $T' \leq T^\epsilon$ for all $\epsilon < \epsilon_0$. The same computations that led to (1.2.38) then imply

$$A(t) \leq A(0) \exp\left(\int_0^t \frac{2CA\epsilon}{\langle s \rangle} \, \mathrm{d}s \right) \leq A\epsilon \exp\left(\int_0^t \frac{2CA\epsilon}{\langle s \rangle} \, \mathrm{d}s \right). \tag{1.2.41}$$

We now use the explicit form of T^ϵ to evaluate the integral

$$\begin{aligned} \int_0^t \frac{2CA\epsilon}{\langle s \rangle} \, \mathrm{d}s \leq 2CA\epsilon \int_0^{T^\epsilon} \frac{1}{\langle s \rangle} \, \mathrm{d}s &= 2CA\epsilon \int_0^{T^\epsilon} \frac{1}{\sqrt{1+s^2}} \, \mathrm{d}s \\ &= 2CA\epsilon \sinh^{-1}(t) \Big|_{t=0}^{t=T^\epsilon} = 2CA\epsilon \sinh^{-1}\left(\sinh\left(\frac{B}{\epsilon} \right) \right) \\ &= 2CAB. \end{aligned}$$

We conclude from these computations and our choices of B, ϵ_0 that

$$A(t) \leq A\epsilon \exp(2CAB) < A\epsilon$$

for all $0 \leq t \leq T'$. This concludes the improvement of the bootstrapping assumption, (b), and the theorem. □

1.3 Main results of the current work

We now give a very brief summary of how the results of dissertation tie into a rapidly advancing research program of wave-type equations. In Subsections 1.3.2 and 1.3.3, we give a slightly more detailed overview of the results; we refer the reader to Chapters 3 and 4 for further details. The reader should keep in mind that the key a priori estimates needed for the analysis are proved using the vector field method.

1.3.1 Overview

The current work uses the commuting vector field method to analyze the long-time behavior of solutions to nonlinear hyperbolic partial differential equations (PDEs). Our approach is to take a non-trivial symmetric "background" solution, add a symmetry-breaking perturbation, and study its dynamical stability on the whole space-time. Here "stability" of the background is quantified by deriving uniform decay estimates for the perturbations, showing that the inherent symmetries survive in the long-time asymptotics.

The classical analytic framework for wellposedness of general equations views the solutions to the Cauchy initial value problem as perturbations of a trivial constant background solution [Hör97, Tao06, Sog08]. The recent breakthrough works of Christodoulou [Chr07, Chr09] have led to new analytical techniques that can account for non-trivial symmetric background solutions. For example, in [Chr09], he takes a *spherically symmetric* solution to the Einstein field equations which forms a black hole (which he proved exists in [Chr91]), and adds to it a *symmetry-breaking* perturbation. He is able to get quantitative control of the perturbation by exploiting special structures (or *null conditions* in the literature) of the equations which cancel harmful terms that would compromise the dynamics. With this control, he is able to show that the lower dimensional behavior is stable for subsequent times.

In a similar vein, the stability properties of some affine totally geodesic maps from Minkowski space to a spaceform are studied in Chapter 3. Viewing our background map as a one dimensional ODE solution of the **wave map equation**, we perturb the homogeneity to prove:

Theorem 1.3.1 (Very rough version of Theorem 1.3.5)**.** *Totally geodesic maps from Minkowski space $\mathbb{R}^{1,3}$ to a spaceform (M,g) are globally stable solutions to the wave map equation.*

Using the spaceform structures of the target, we prove that the equations of motion for the perturbation decouple into a nonlinear system of wave-Klein–Gordon equations satisfying a certain weak null condition. Using the technical weighted Sobolev interpolation estimates proved in Chapter 2, we prove global existence for this system and derive uniform decay properties of the perturbation, showing that the homogeneous ODE picture is stable.

Quantitative control of perturbations by exploiting null conditions is also the strategy taken by Christodoulou in his monograph [Chr07]. There he proves stability of a background solution of the compressible Euler equations that exhibits shock singularities akin to the $(1+1)$-dimensional Burgers' equation. By the 1970s [Lax64, Lax73, Joh74], this Burgers-type shock singularity was known to occur for planewave[2] solutions to equations satisfying Lax's resonant genuinely nonlinear condition. The shock mechanism has recently been shown [SHLW16] to be a stable phenomenon in the above sense for such equations. In contrast, *global*-in-time planewave solutions have been constructed for the relativistic **membrane equation** [Lin04, Won17a]. This was possible because the membrane equation enjoys extremely strong null conditions which cancel those terms that would exhibit the resonant condition of Lax, precluding singular behavior. One can similarly ask whether the global results of this lower dimensional phenomena are stable. The main result of Chapter 4, adapted from original joint work [AW19a], answers this question in the affirmative. We show:

Theorem 1.3.2 (Very rough version of Theorem 1.3.6)**.** *Simple planewave solutions to the membrane equation are globally stable under symmetry-breaking perturbations.*

Remark 1.3.3 (The intimate role played by null conditions)**.** Klainerman [Kla86] and Christodoulou [Chr86] independently discovered a sufficient algebraic condition on the nonlinearities of wave equations on $\mathbb{R}^{1+3}$ that guarantees global-in-time existence for sufficiently small data. A physical interpretation of the null condition is that it prevents nonlinear interactions between wave packets traveling along the same null geodesic. That

[2] Planewave solutions are those whose dynamics are only nontrivial in one direction.

this has an effect on the large-time behavior makes sense in view of the fact that wave packets that are localized around the null geodesic can interact for an arbitrary amount of time, in contrast to those which are traveling in different directions; *this latter behavior is guaranteed by the classical null condition of Klainerman and Christodoulou [Kla86,Chr86].*

The equations studied in this dissertation (wave maps, membrane equation) both satisfy the classical null condition. However, since we are considering initial perturbations of *large* background solutions, we can no longer appeal to the small-data results of [Kla86, Chr86]. Instead, crucial to our analysis is the existence of "vestigial" null conditions" that survive the perturbations, see the discussions surrounding (1.3.4) and (1.3.9).

Remark 1.3.4 (Physical Relevance)**.** The equations studied in this dissertation describe the dynamics of physical models (i.e. membranes [Neu90, Jer11], nonlinear sigma models [GML60,CBM96], etc). Historically, mathematicians sought to provide rigorous mathematical analysis of these physical models by restricting their attention to *symmetrically reduced* solutions. Although symmetric regimes are of mathematical interest in themselves, the extent of their applications is limited in that physical phenomena is never truly symmetric. As is evident by the monumental works describing the stability of Minkowski space [CK93, LR05, BZ09], shock formation in compressible fluids [Chr07, CM14, LS18], and formation of black holes [Chr09, AL14], there is an extensive gap in difficulty in passing from the case of symmetric solutions to those without symmetry. Hence, this dissertation serve to add to this rapidly advancing and physically relevant sub-field of PDEs.

1.3.2 Totally geodesic wave maps

A map $\phi : N \to M$ between two pseudo-Riemannian manifolds (N, h) and (M, g) is said to be *totally geodesic* if it maps geodesics in N to geodesics in M. This is characterized by the total vanishing of the *fundamental form* of ϕ, namely the tensor $\nabla^* \mathrm{d}\phi$. Here, ∇^* is the pull-back of the Levi-Civita connection ∇^M onto N. An immediate consequence of this is that totally geodesic maps are critical points of the energy functional

$$\mathcal{S}[\phi] = \int_N \langle \mathrm{d}\phi, \mathrm{d}\phi \rangle_{T^*M \otimes \phi^* TM} \ \mathrm{dvol}_h$$

because its Euler-Lagrange equations (ELE) are precisely $\mathrm{tr}_h \nabla^* \mathrm{d}\phi = 0$. Setting the domain to be $(\mathbb{R}^{1,d}, \mathfrak{m})$ and writing the ELE in local coordinates on the target manifold M, we see that totally geodesic maps automatically satisfy the *wave maps equation*

$$\Box_{\mathfrak{m}} \phi^i + \Gamma^k_{ij}(\phi) \langle \phi^i, \phi^j \rangle_{\mathfrak{m}} = 0. \tag{1.3.1}$$

The wave maps equation simultaneously generalize the geodesic and linear wave equations, as can be seen directly from (1.3.1) and setting $d = 0$ or $M = \mathbb{R}$, respectively.

In the Riemannian setting, Vilms [Vil70] characterized totally geodesic maps between Riemannian manifolds as a composition of a Riemannian submersion followed by a Riemannian immersion, both totally geodesic:

$$\begin{array}{ccc} N & \xrightarrow{\quad\Phi\quad} & M. \\ & {}_{\Phi_{\mathrm{S}}}\searrow \quad \nearrow_{\Phi_{\mathrm{I}}} & \\ & B & \end{array}$$

With this as a motivational starting point, our work serves to expand on the literature of totally geodesic maps in the Lorentzian regime by analyzing the dynamical stability of mappings that factor as

$$\mathbb{R}^{1+d} \xrightarrow{\varphi_{\mathrm{S}}} \mathbb{R} \xrightarrow{\varphi_{\mathrm{I}}} M; \tag{1.3.2}$$

where, denoting by e the standard Euclidean metric on $\mathbb{R}$, the mapping φ_{S} is a semi-Riemannian submersion to either $(\mathbb{R}, e)$ or $(\mathbb{R}, -e)$, and φ_{I} is a Riemannian immersion from $(\mathbb{R}, e)$ to a spaceform (M, g). In particular, this factorization implies the background solution is automatically a totally geodesic wave map that has infinite total energy. The semi-Riemannian submersion φ_{S} can be classified as *spacelike* or *timelike*[3] depending on

[3]Note that by definition, a semi-Riemannian submersion cannot be null. We always equip the real line $\mathbb{R}$, as the domain of φ_{I}, with $+e$.

whether its codomain $\mathbb{R}$ is considered as being equipped with e or $-e$. A rough version of our theorem can be stated as

Theorem 1.3.5 (Rough version of Theorems 3.6.1 and 3.7.1)**.** *Fix $d \geq 3$. A totally geodesic map satisfying the factorization* (1.3.2) *is globally nonlinearly stable as a solution to the initial value problem for the wave maps equation under compactly supported smooth perturbations, provided that either*

TL: *φ_S is timelike and (M,g) is a negatively-curved spaceform, or*

SL: *φ_S is spacelike and (M,g) is a positively-curved spaceform.*

We derive the equations of motion for the perturbation in a tubular neighborhood $\mathbb{R} \times \mathcal{N}$ of the geodesic $\varphi_{\mathrm{I}}(\mathbb{R}) \subset M$ (here $\mathbb{R}$ parametrizes the geodesic and $\mathcal{N}$ the normal $(n-1)$-directions). The main geometric contribution of Chapter 3 is to show that the equations for the perturbation $\boldsymbol{u} = (u^1, \vec{u}) \in \mathbb{R} \times \mathcal{N}$ decouple into a system of wave and Klein–Gordon equations:

$$\begin{cases} \Box u^1 = F^1 \boldsymbol{u} \cdot \langle \mathrm{d}\boldsymbol{u}, \mathrm{d}\varphi_{\mathrm{S}} \rangle_{\mathfrak{m}} + O(|\boldsymbol{u}|, |\partial \boldsymbol{u}|^3), \\ \Box \vec{u} - \vec{M} \vec{u} = \vec{F} \boldsymbol{u} \cdot \langle \mathrm{d}\boldsymbol{u}, \mathrm{d}\varphi_{\mathrm{S}} \rangle_{\mathfrak{m}} + O(|\boldsymbol{u}|, |\partial \boldsymbol{u}|^3). \end{cases} \tag{1.3.3}$$

Here F^1, $\vec{F}$ are functions of the curvature of (M,g) restricted to the geodesic φ_{I}. The $\vec{M}$ are the masses of $\vec{u}$, and as a consequence of the spaceform assumption on M, we prove that $\vec{M} = \kappa \langle \mathrm{d}\varphi_{\mathrm{S}}, \mathrm{d}\varphi_{\mathrm{S}} \rangle_{\mathfrak{m}}$ where κ is the sectional curvature of M. Hence, the assumptions on φ_{S} in Theorem 1.3.5 are there to at minimum guarantee linear stability, i.e. make the Klein–Gordon terms $\vec{u}$ have positive masses.

The computations leading to (1.3.3) hinge on a careful Taylor expansion of the Christoffel symbols Γ in the wave maps equation (1.3.1) about the geodesic $\varphi_{\mathrm{I}}(\mathbb{R})$. This is where we utilize the spaceform assumption on M; it forces certain cancellations of these Taylor coefficients which reveal hidden null-structures of the equations. More concretely, we prove that the nonlinearities of (1.3.3) can be written schematically as

$$\vec{u} \cdot \langle \boldsymbol{u}, \mathrm{d}\varphi_{\mathrm{S}} \rangle_{\mathfrak{m}} + \vec{u} \cdot O(|\boldsymbol{u}|^2 + |\partial \boldsymbol{u}|^2). \tag{1.3.4}$$

From this structure we see a complete vanishing of resonant wave–wave interactions that could lead to finite-time blow up. Indeed, as we are unable to use the Morawetz vector field as a multiplier, the available decay rate for the undifferentiated wave u^1 in dimension 3 is $t^{-1/2}$; the exposed null structures serve to completely remove deadly terms of the form $(u^1)^2$ or $(u^1)^3$. Secondly, notice that every term has as a factor a Klein–Gordon solution $\vec{u}$. This is crucially important because the linear decay rate for the Klein–Gordon equation is the *integrable* power $t^{-3/2}$, which serves to dampen the nonlinear feedback and allow for a global solution. We note, however, that this is the extent of the improved decay which we can extract from (1.3.4). This is because the background φ_{S}, as a consequence of being a semi-Riemannian submersion, *cannot be null*. Its interaction with the perturbation $\langle \mathrm{d}\boldsymbol{u}, \mathrm{d}\varphi_{\mathrm{S}} \rangle_{\mathfrak{m}}$ is then a *generic derivative* $\partial \boldsymbol{u}$ and not a *tangential* derivative for which there would be an improved decay rate.

1.3.3 Membranes

The starting point of our discussion is the equation

$$\frac{\partial}{\partial x^\mu}\left(\frac{\mathfrak{m}^{\mu\nu}\partial_\nu \phi}{\sqrt{1 + \mathfrak{m}(\nabla\phi, \nabla\phi)}} \right) = 0 \tag{1.3.5}$$

on $\mathbb{R}^{1,d}$, the $(1+d)$ dimensional Minkowski space equipped with the metric $\mathfrak{m}$ which in standard coordinates is given by the diagonal matrix $\mathrm{diag}(-1, 1, 1, \cdots, 1)$. In the equation we used the notation $\mathfrak{m}(\nabla\phi, \nabla\psi) \stackrel{\text{def}}{=} \mathfrak{m}^{\mu\nu}\partial_\mu\phi\partial_\nu\psi$. This equation is variously known as the *membrane equation*, the *timelike minimal/maximal surface equation*, or the *Lorentzian vanishing mean curvature flow*. This is due to the interpretation that the graph of ϕ in $\mathbb{R}^{1,d} \times \mathbb{R} \cong \mathbb{R}^{1,d+1}$ is an embedded timelike hypersurface with zero mean curvature.

Solutions to (4.1.1) model extended test objects (world sheets), in the sense that the case where $d = 0$ reduces to the geodesic equation which models the motion of a test particle. (The membrane equation can also be formulated with codimension greater than one; see [AAI06, Mil08].) The membranes can also interact with external forces which manifests as a prescription of the mean curvature; see [AC79, Hop13, Kib76, VS94] for some discussion of the physics surrounding such objects, and see [Jer11, Neu90] for rigorous justifications that membranes represent extended particles.

The precise version of our main theorem is Theorem 4.5.8; there we state the result as a small-data global existence result for the corresponding perturbation equations, after a nonlinear change of independent variables that corresponds to a gauge choice. Here we state a slightly less precise version in terms of the original variables.

Theorem 1.3.6 (Rough version of Theorem 4.5.8)**.** *Fix the dimension $d = 3$. Let Υ denote a smooth simple-plane-symmetric solution to* (4.1.1) *with finite extent in its direction of travel. Fix a bounded set $\Omega \subset \mathbb{R}^3$. There exists some $\epsilon_0 > 0$ depending on the background Υ and the domain Ω, such that for any $(\psi_0, \psi_1) \in (H^5(\mathbb{R}^3) \cap C_0^\infty(\Omega)) \times (H^4(\mathbb{R}^3) \cap C_0^\infty(\Omega))$ with $\|(\psi_0, \psi_1)\| < \epsilon_0$, the initial value problem to* (4.1.1) *with initial data*

$$\phi(0,x) = \Upsilon(0,x) + \psi_0(x), \qquad \partial_t \phi(0,x) = \partial_t \Upsilon(0,x) + \psi_1(x)$$

has a global solution that converges in $C^2(\mathbb{R}^3)$ to Υ as $t \to \pm\infty$.

Here Υ is to be interpreted as a traveling "pulse"; it is a compactly supported function that is both constant in two of the spatial variables *and* one whose differential $\mathrm{d}\Upsilon$ is null with respect to the dynamic metric. This has the physical interpretation that Υ propagates along only one of the characteristic directions of the nonlinear wave equation. We remark that our results extend to $d \geq 3$, with adjustments made to the regularity assumptions on the initial data.

The equation of motion for ϕ, upon dropping higher order terms momentarily, is effectively

$$\Box_{\mathfrak{m}} \phi + \phi \Upsilon''(\partial_t + \partial_{x^1})^2 \phi + \Upsilon''(\partial_t \phi + \partial_{x^1} \phi)^2 = 0. \tag{1.3.6}$$

Here $\Box_{\mathfrak{m}} = -\partial_t^2 + \Delta_x$, and the background pulse is assumed to be traveling in the $+x^1$ direction, so has compact support in the $(t - x^1)$ variable. This is derived by a convenient choice of gauge where the perturbation is written as a graph in the normal bundle of Υ, interpreted as a submanifold of $\mathbb{R}^{1,d+1}$. This gauge is also used in [DKSW16], where the authors studied the stability problem for the static catenoid solutions to the membrane equation. Global existence for equations satisfying a verison of the null condition is known in spatial dimensions two and three [Kla80,Kla82,Kla84,Chr86,Ali01a,Ali01b]. However, the presence of Υ'' as a coefficient of the resonant terms means that one can't directly apply the classical null condition arguments because it is not Lorentz invariant. More sinister still is the fact that these coefficients have a *growing weight* when differentiated by the Lorentz boosts (which are the weighted commutator fields mentioned in the previous section adapted to the geometry of this problem):

$$L^i \Upsilon'' = (t\partial_{x^i} + x^i \partial_t)\Upsilon'' = (x^i - \delta^{1i} t)\Upsilon'''. \tag{1.3.7}$$

This growing weight has the physical interpretation of a transfer of energy from the "infinite energy" background Υ to the perturbation.

This growing weight requires us to use a modified bootstrap argument where the energies of order 2 (controlling 3 derivatives) and higher are allowed to grow polynomially. This is in stark contrast to generic quadratically resonant settings where only the *top* order energies are allowed to grow; the more derivatives are used, the more energies remain bounded.

We begin the analysis of (4.1.2) by dropping the quasilinearity and studying the semilinear model problem

$$\Box_{\mathfrak{m}} \phi + \Upsilon''(\partial_t \phi + \partial_{x^1} \phi)^2 = 0. \tag{1.3.8}$$

This simplified problem encapsulates *most* of the difficulties present in the energy estimates and hence sheds a considerable amount of insight on how to handle the full quasilinear problem. That we are able to close the bootstrap in spite of the growing weights is due to the fact that Υ'' has compact support in the $(t-x^1)$ variable. Since the resonant interacting terms $(\partial_t\phi + \partial_{x^1}\phi)$ represent waves traveling in directions transverse to the level sets of $t-x^1$, the resonant interaction only takes place for a finite amount of time.

When handling the quasilinear problem (4.1.2) the second term

$$\phi\Upsilon''(\partial_t + \partial_{x^1})^2\phi \tag{1.3.9}$$

would naively lose a full derivative (and, due to the growth hierarchy, also lose the associated decay) and consequently the bootstrap would not close. What allows the argument to go through is a "vestigial" null condition in the equation: note that when $i=1$ equation (1.3.7) and the compact support imply that the weight is *not* growing after commuting with the boost L^1. From this we show that $(\partial_t + \partial_{x^1})^2\phi$ decays *faster* than a generic tangential second derivative, and so we can close the auxiliary bootstrap assumptions by examining the system of equations satisfied by ϕ and $L^1\phi$.

CHAPTER 2

THE LINEAR WAVE AND KLEIN–GORDON EQUATIONS

In this chapter we apply the vector field method introduced in the pervious chapter to obtain decay estimates for the linear wave and Klein–Gordon equations. Since the well-posedness theory for these equations is standard [Hör97, Sog08], we focus exclusively on obtaining the decay estimates through the vector field method approach. As we mentioned in the pervious chapter, L^1–L^∞ dispersive decay is typically derived from oscillatory integral control of the explicit Fourier representations of the fundamental solutions [Tao06]. The physical space nature of the vector field method is then better suited for the quasilinear regime of Chapter 4, where the principal symbol of the equation depends on the solution itself.

This chapter begins with Section 2.1, where we first record Morrey-type L^2–L^∞ Sobolev embedding estimates adapted to hyperboloidal foliations of Minkowski space[1]. We also derive interpolated GNS-type L^p–L^q Sobolev embedding estimates, also adapted to the hyperboloidal foliations. These estimates are valid for any scalar function of Schwarz class (and by density arguments for less regular functions), and are not confined to solutions of PDEs. The latter estimates are useful because they allow one to save a derivative over the former ones. A poignant example occurs on $\mathbb{R}^2$. Using only the L^∞ type Sobolev estimates we can bound

$$\|\phi^2\|_{L^2(\mathbb{R}^2)} \le \|\phi\|_{L^\infty(\mathbb{R}^2)}\|\phi\|_{L^2(\mathbb{R}^2)} \lesssim \|\phi\|_{H^2(\mathbb{R}^2)}\|\phi\|_{L^2(\mathbb{R}^2)}.$$

(Scaling would have given us the first factor of ϕ in H^1, but as we know the end-point Sobolev embedding in L^∞ is false.) Using L^p type Sobolev inequalities instead we can

[1] For motivation on why a hyperboloidal foliation approach is taken, see the introductory discussion of Section 2.2 and Remark 2.2.6

appeal to Ladyzhenskaya's inequality to get

$$\|\phi^2\|_{L^2(\mathbb{R}^2)} \lesssim \|\phi\|_{H^1(\mathbb{R}^2)} \|\phi\|_{L^2(\mathbb{R}^2)}$$

for a gain of one derivative.

The main results of this chapter are found in Section 2.2. There, the Morrey- and GNS-type estimates are used *in conjunction with the vector field method* to derive a priori estimates for solutions to the linear wave and Klein–Gordon equations. More specifically, the former are pointwise decay estimates (see Proposition 2.2.7) whereas the latter are a family of integrated decay estimates (see Subsections 2.2.2 and 2.2.3).

We point out that the Morrey-type estimates (and their associated pointwise decay estimates for the linear wave and Klein–Gordon equations) are not original work. They originated in Le Floch and Ma's work [LM14] (see also Wong [Won17b]), which adapted Klainerman's foundational vector field method [Kla85b] to derive L^2–L^∞ estimates for the wave equation adapted to hyperboloidal foliations of Minkowski space. Hence, we restrict our analysis to rigorously deriving the GNS-type estimates in Section 2.1 and their associated integrated decay for solutions of the linear wave and Klein–Gordon equations in Subsections 2.2.2 and 2.2.3. This is original work based on [AW19b], and whose results can be viewed as the counterpart to the Morrey theory extended to weighted L^p based Sobolev spaces. We chose to include the Morrey-type results in this chapter because they play a crucial role in the arguments used in Chapters 3 and 4. Moreover, since the recurring theme of the present work is to showcase the vector field method, we highlight and overview the role it plays in deriving the a priori estimates of Section 2.2. For the convenience of the reader, complete proofs of the technical details based on Wong's geometric formulation [Won17b] are provided in Appendix A.2.

2.1 Sobolev embeddings adapted to hyperboloidal foliations

Keeping in mind the expectation that these integrals will be viewed as being adapted to a hyperboloidal foliation, we will set our notation accordingly. By Σ_τ we refer to the hyperboloid in $\mathbb{R}^{1+d}$ given by

$$\Sigma_\tau \stackrel{\text{def}}{=} \{t^2 - |x|^2 = \tau^2, t > 0\}. \tag{2.1.1}$$

We can parametrize it by $\mathbb{R}^d$ via the map

$$(x^1, \dots, x^d) \mapsto (t = \sqrt{\tau^2 + |x|^2}, x^1, \dots, x^d) \in \mathbb{R}^{1+d}. \tag{2.1.2}$$

For convenience throughout we will denote by

$$w_\tau(x) \stackrel{\text{def}}{=} \sqrt{\tau^2 + |x|^2}, \quad x \in \mathbb{R}^d. \tag{2.1.3}$$

We note that the value of w_τ, when thinking of Σ_τ as embedded in $\mathbb{R}^{1+d}$, of course agrees with the value of the t coordinate; we use the notation w_τ as mental aid to work intrinsically on Σ_τ whenever appropriate.

The Minkowski metric on $\mathbb{R}^{1+d}$ induces a Riemannian metric on Σ_τ, which is given by the matrix-valued function

$$h_{ij} = \delta_{ij} - \frac{x^i x^j}{w_\tau(x)^2} \tag{2.1.4}$$

relative to the parametrization above. This being a rank-one perturbation of the Euclidean metric, the corresponding volume form can be easily computed to be

$$\mathrm{dvol}_\tau = \frac{\tau}{w_\tau} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d. \tag{2.1.5}$$

The Minkowski space $\mathbb{R}^{1+d}$ admits as Killing vector fields the Lorentzian boosts, given as

$$L^i \stackrel{\text{def}}{=} x^i \partial_t + t \partial_{x^i}. \tag{2.1.6}$$

These vector fields are *tangent* to the hypersurfaces Σ_τ for every $\tau > 0$ and span the tangent space at every point. In the parametrization above they can be identified with

$$L^i \cong w_\tau \partial_{x^i}. \tag{2.1.7}$$

We remark that

$$L^i w_\tau = x^i, \quad L^i x^i = w_\tau.$$

In particular, we have that for any string of derivatives

$$\left|L^{i_1}\cdots L^{i_K} w_\tau\right| \le w_\tau. \tag{2.1.8}$$

If α is an m-tuple with elements drawn from $\{1,2,3\}$ (namely that $\alpha = (\alpha_1,\dots,\alpha_m)$ with $\alpha_i \in \{1,2,3\}$) we denote

$$L^\alpha u \stackrel{\text{def}}{=} L^{\alpha_m} L^{\alpha_{m-1}} \cdots L^{\alpha_1} u. \tag{2.1.9}$$

By $|\alpha|$ we refer to its length, namely m.

Almost all of the analysis of this work uses the following weighted Lebesgue and Sobolev spaces:

- For $p \in [1,\infty)$ and $\alpha \in \mathbb{R}$, by $\mathcal{L}^p_\alpha$ we refer to the weighted Lebesgue norm

$$\|u\|_{\mathcal{L}^p_\alpha} = \left(\int_{\Sigma_\tau} w_\tau^\alpha |u|^p \ \mathrm{dvol}_\tau \right)^{1/p}. \tag{2.1.10}$$

- For $p \in [1,\infty)$, $\alpha \in \mathbb{R}$, and $k \in \mathbb{N}$, by $\mathring{\mathcal{W}}^{k,p}_\alpha$ we refer to the weighted homogeneous Sobolev norm

$$\|u\|_{\mathring{\mathcal{W}}^{k,p}_\alpha} = \sum_{|\beta|=k} \|L^\beta u\|_{\mathcal{L}^p_\alpha}. \tag{2.1.11}$$

  The corresponding inhomogeneous version $\mathcal{W}^{k,p}_\alpha$ is

$$\|u\|_{\mathcal{W}^{k,p}_\alpha} = \sum_{j=0}^{k} \|u\|_{\mathring{\mathcal{W}}^{j,p}_\alpha}. \tag{2.1.12}$$

The main L^2–L^∞ estimates along the hyperboloids Σ_τ are

Theorem 2.1.1. ([Won17b, Theorem 2.18]). *Let $\ell \in \mathbb{R}$ be fixed. Then the following uniform estimate holds for functions u defined on the set $\{t > |x|\} \subset \mathbb{R}^{1+d}$ (with the implicit constant depending only on d and ℓ):*

$$\tau^{1/2} \left\| w_\tau^{(d+l-1)/2} \cdot u \right\|_{L^\infty(\Sigma_\tau)} \lesssim \|u\|_{\mathcal{W}^{\lfloor d/2 \rfloor + 1, 2}_\ell}. \tag{2.1.13}$$

Proof. See appendix A.2.2. □

With regards to the general approach to the vector field method described in the Prologue of Chapter 1, estimate (2.1.13) sets up the "pointwise estimates from higher derivative integral norms" described in Point **III**.

2.1.1 The basic global GNS inequalities

The Nirenberg argument [Nir59] is built upon the fundamental theorem of calculus. Given a point $x \in \mathbb{R}^d$, we will write

$$x_i'(s) \stackrel{\text{def}}{=} (x^1, x^2, \ldots, x^{i-1}, s, x^{i+1}, \ldots, x^d)$$

as the point where the ith coordinate of x is replaced by the real parameter s. Then the fundamental theorem of calculus states that, for any smooth, compactly supported function u,

$$|u(x)| \leq \int_{-\infty}^{x^i} |\partial_i u(x_i'(s))| \, \mathrm{d}s \leq \int_{-\infty}^{\infty} \frac{1}{w_\tau \circ x_i'(s)} |L^i u(x_i'(s))| \, \mathrm{d}s. \tag{2.1.14}$$

This implies

$$|u(x)|^{\frac{d}{d-1}} \leq \prod_{i=1}^{d} \left(\int_{\mathbb{R}} \frac{|L^i u(x_i'(s))|}{w_\tau \circ x_i'(s)} \, \mathrm{d}s \right)^{\frac{1}{d-1}}. \tag{2.1.15}$$

Now, integrating the left hand side and applying Hölder's inequality (exactly as in [Nir59]) this implies (noting that the volume form is weighted according to (2.1.5))

$$\tau^{\frac{1}{d-1}} \int_{\Sigma_\tau} w_\tau(x) |u(x)|^{\frac{d}{d-1}} \, \mathrm{dvol}_\tau \leq \prod_{i=1}^{d} \left(\int_{\Sigma_\tau} |L^i u(x)| \, \mathrm{dvol}_\tau \right)^{\frac{1}{d-1}}. \tag{GNS$_1$}$$

The extra factor of τ comes from the dvol that appears different number of times on the two sides. Taking advantage of (2.1.8) which shows that we have really an exponential-type weight, (GNS$_1$) implies the following arbitrarily-weighted counterpart. For any $\alpha \in$

$\mathbb{R}$,

$$\tau^{\frac{1}{d-1}} \int_{\Sigma_\tau} w_\tau^{1+\alpha\cdot\frac{d}{d-1}} |u(x)|^{\frac{d}{d-1}} \,\mathrm{dvol}_\tau \le \prod_{i=1}^{d} \left(\int_{\Sigma_\tau} w_\tau^\alpha |L^i u| + |\alpha| w_\tau^\alpha |u| \,\mathrm{dvol}_\tau \right)^{\frac{1}{d-1}}. \quad (\mathrm{GNAWS}_1)$$

(This last inequality follows by replacing $u \mapsto w_\tau^\alpha u$ in (GNS_1).)

So (GNAWS_1) asserts the continuous embedding $\mathcal{W}_\alpha^{1,1} \hookrightarrow \mathcal{L}_{\alpha d/(d-1)+1}^{d/(d-1)}$.

Remark 2.1.2. To foreshadow our discussion, notice that the standard ∂_t-energy of the linear wave equation (see (2.2.10) and Lemma 2.2.5) controls

$$\tau^{-1}\|u\|^2_{\mathring{\mathcal{W}}^{1,2}_{-1}} + \tau\|\partial_t u\|^2_{\mathcal{L}^2_{-1}}.$$

On the other hand, the ∂_t-energy of the linear Klein–Gordon equation controls

$$\tau^{-1}\|u\|^2_{\mathring{\mathcal{W}}^{1,2}_{-1}} + \tau\|\partial_t u\|^2_{\mathcal{L}^2_{-1}} + \tau^{-1}\|u\|^2_{\mathcal{L}^2_{1}}$$

(note the different weight on the final term).

Replacing u by u^q, coupled with an application of Hölder's inequality, gives the standard extensions of (GNS_1) and (GNAWS_1) to $\mathcal{W}_\alpha^{1,p}$. Let $1 \le p < d$, we have

$$\tau^{1/d}\|u\|_{\mathcal{L}_1^{dp/(d-p)}} \lesssim \|u\|_{\mathring{\mathcal{W}}^{1,p}_{1-p}}, \quad (\mathrm{GNS_p})$$

$$\tau^{1/d}\|u\|_{\mathcal{L}^{dp/(d-p)}_{1+\alpha dp/(d-p)}} \lesssim \|u\|_{\mathcal{W}^{1,p}_{1-p+\alpha p}}. \quad (\mathrm{GNAWS_p})$$

Iterating ($\mathrm{GNAWS_p}$) above, we also have as a corollary that, given $k \in \mathbb{N}$ and $p \in [1,\infty)$ such that $kp < d$, for any $\beta \in \mathbb{R}$,

$$\tau^{k/d}\|u\|_{\mathcal{L}^q_{1-q+q(\beta+k)}} \lesssim \|u\|_{\mathcal{W}^{k,p}_{1-p+p\beta}}, \quad (\mathrm{GNAWS_{pk}})$$

where $q = dp/(d-kp)$ is the usual Sobolev conjugate of p. We note that the case $\beta + k = 1$ is essentially a re-formulation of the standard Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}^d$.

Remark 2.1.3. Notice that formally setting $p = 2$, $k = d/2$, and $\beta = 0$, one sees that ($\mathrm{GNAWS_{pk}}$) has the correct scaling for an inequality of the type

$$\tau^{1/2}\|w_\tau^{d/2-1}u\|_{L^\infty} \quad \text{“}\lesssim\text{”} \quad \|u\|_{\mathcal{W}_{-1}^{d/2,2}}.$$

This inequality, as we know, is not true, due to the failure of the end-point Sobolev inequality into L^∞. On the other hand, the (Morrey-type) global Sobolev inequality as stated in Theorem 2.1.1 can be restated in the following form

$$\tau^{1/2}\|w_\tau^{d/2-1}u\|_{L^\infty} \lesssim \|u\|_{\mathcal{W}_{-1}^{\lfloor d/2\rfloor+1,2}}. \tag{2.1.16}$$

2.1.2 Interpolating inequalities: non-borderline case

The inequalities ($\mathrm{GNS_p}$) and ($\mathrm{GNAWS_p}$) represent the endpoint Sobolev embeddings, when $p < d$, in our setting. In this Subsection we prove Gagliardo-Nirenberg type interpolation inequalities. For simplicity we will focus on the case of one derivative: that is, we examine embeddings of the form

$$\mathcal{W}_\alpha^{1,p} \cap \mathcal{L}_\beta^q \hookrightarrow \mathcal{L}_\gamma^r$$

with $q \le r \le dp/(d-p)$. The case of higher derivatives, based on ($\mathrm{GNAWS_{pk}}$), is analogous and left to the reader. For convenience we denote $p^* \stackrel{\text{def}}{=} \frac{dp}{d-p}$ as the Sobolev conjugate of p.

Proposition 2.1.4. *Given $q \le r \le p^*$, and let $\theta \in [0,1]$ satisfy*

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p^*}.$$

Then the following inequalities hold for any $\alpha, \beta \in \mathbb{R}$:

$$\tau^{(1-\theta)/d}\|u\|_{\mathcal{L}^r_{1+\theta\beta r}} \lesssim \left(\|u\|_{\mathcal{L}^q_{1+\beta q}}\right)^\theta \cdot \left(\|u\|_{\mathring{\mathcal{W}}^{1,p}_{1-p}}\right)^{1-\theta}, \tag{$\mathrm{GNS_{pqr}}$}$$

$$\tau^{(1-\theta)/d}\|u\|_{\mathcal{L}^r_{1+(\theta\beta+(1-\theta)\alpha)\cdot r}} \lesssim \left(\|u\|_{\mathcal{L}^q_{1+\beta q}}\right)^\theta \cdot \left(\|u\|_{\mathcal{W}^{1,p}_{1-p+\alpha p}}\right)^{1-\theta}. \tag{$\mathrm{GNAWS_{pqr}}$}$$

Proof. The inequalities hold by applying the following elementary interpolation inequality of the weighted $\mathcal{L}^p_\alpha$ spaces: for all $\theta \in [0,1]$,

$$\|u\|_{\mathcal{L}^r_{\beta\theta+(1-\theta)\alpha}} \leq \|u\|^\theta_{\mathcal{L}^q_{\beta q/r}} \cdot \|u\|^{1-\theta}_{\mathcal{L}^p_{\alpha p/r}}, \tag{2.1.17}$$

whenever

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}.$$

□

2.1.3 Interpolating inequalities: borderline case

In the previous Subsection we treated the interpolation inequalities when $p < d$. In this Subsection we treat the interpolation inequalities when $p = d$. Specifically, we examine embeddings of the form

$$\mathcal{W}^{1,d}_\alpha \cap \mathcal{L}^q_\beta \hookrightarrow \mathcal{L}^r_\gamma$$

where now $1 \leq q \leq r < \infty$. In view of our applications, the case $p = d = 2$ will be of specific interest. We occasionally abbreviate the Sobolev conjugate $1^* = d/(d-1)$.

Proposition 2.1.5. *Let $q \leq r < \infty$, and $\beta \in \mathbb{R}$. Then*

$$\left(\tau^{1/d}\right)^{\frac{r-q}{r}} \|u\|_{\mathcal{L}^r_{1+\theta\beta r}} \lesssim \left(\|u\|_{\mathcal{L}^q_{1+\beta q}}\right)^{q/r} \cdot \left(\|u\|_{\mathring{\mathcal{W}}^{1,d}_{(1-d)(1+\beta\theta r)}}\right)^{(r-q)/r}, \tag{GNS$_{\mathrm{dqr}}$}$$

where $\theta \in (0,1]$ is the solution to

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{r+1^*}.$$

Proof. Replacing $u \mapsto u^{1+r/1^*}$ in (GNS_1) implies

$$\begin{aligned}
\tau^{1/d}\left(\int w_\tau |u|^{r+1^*} \,\mathrm{dvol}\right)^{1/1^*} &\lesssim \sum_{i=1}^d \int |u|^{r/1^*} |L^i u| \,\mathrm{dvol} \\
&\lesssim \left(\int w_\tau^{1+\theta\beta r} |u|^r \,\mathrm{dvol}\right)^{1/1^*} \cdot \|u\|_{\mathring{\mathcal{W}}^{1,d}_{(1-d)(1+\beta\theta r)}}
\end{aligned}$$

by Hölder's inequality. Here we used that

$$1 = w_\tau^{(1+\theta\beta r)/1^*} \cdot w_\tau^{(-1-\theta\beta r)/1^*}.$$

This in particular implies

$$\tau^{1/(d-1)} \|u\|^{r+1^*}_{\mathcal{L}^{r+1^*}_1} \lesssim \|u\|^r_{\mathcal{L}^r_{1+\theta\beta r}} \|u\|^{1^*}_{\mathring{\mathcal{W}}^{1,d}_{(1-d)(1+\theta\beta r)}}. \tag{2.1.18}$$

We next interpolate using (2.1.17) to find

$$\|u\|_{\mathcal{L}^r_{1+\theta\beta r}} \le \left(\|u\|_{\mathcal{L}^q_{1+\beta q}}\right)^\theta \cdot \left(\|u\|_{\mathcal{L}^{r+1^*}_1}\right)^{1-\theta}.$$

Plugging (2.1.18) in, cancelling the extra factors on both sides, we get the desired inequality after noting that θ is given by

$$\theta = \frac{1^* q}{r(1^* + r - q)}, \qquad 1 - \theta = \frac{(r-q)(r+1^*)}{r(1^*+r-q)}.$$

□

We note that when $\beta = 0$, the triple of weights

$$(1 + \theta\beta r, 1 + \beta q, (1-d)(1+\beta\theta r)) = (1, 1, 1-d).$$

Replacing $u \mapsto w_\tau^\alpha u$ we further have as a corollary

$$\left(\tau^{1/d}\right)^{\frac{r-q}{r}} \|u\|_{\mathcal{L}^r_{1+\theta\beta r + \alpha r}} \lesssim \left(\|u\|_{\mathcal{L}^q_{1+\beta q+\alpha q}}\right)^{q/r} \left(\|u\|_{\mathcal{W}^{1,d}_{(1-d)(1+\beta\theta r)+\alpha d}}\right)^{(r-q)/r}. \tag{GNAWS$_{\mathrm{dqr}}$}$$

2.2 A priori estimates via the vector field method

2.2.1 Energy formalism and pointwise estimates

Our goal is to derive pointwise and integrated decay for a scalar function $\phi : \mathbb{R}^{1+d} \to \mathbb{R}$ which is a solution of

$$-\partial_t^2 \phi + \Delta_x \phi - M^2 \phi = 0, \tag{2.2.1}$$

where $M \in \mathbb{R}$ represents the particle mass. Note that the case where $M \equiv 0$ reduces (2.2.1) to the linear wave equation. We will prescribe initial data at on the slice $t = 0$:

$$\begin{aligned} \phi(0,x) &= f \\ \partial_t \phi(0,x) &= g \end{aligned} \tag{2.2.2}$$

Based on the strategy introduced in the beginning of Chapter 1 (see also the discussion leading to (1.2.26)), we look for a collection of vector fields $\{\mathcal{Z}\}$ which preserve the flow of (2.2.1). That is, if ϕ solves (2.2.1), so does $Z\phi$. It is straightforward to check that all of the vector fields that satisfy this requirement are linear combinations of

- space-time translations: $\partial_t, \partial_{x^1}, \ldots, \partial_{x^d}$;
- spatial rotations: $\Omega_{ij} = x^i \partial_{x^j} - x^j \partial_{x^i}$;
- Lorentz boosts: $L^i = t\partial_{x^i} + x^i \partial_t$.

These vector fields are precisely those that preserve the underlying geometry of Minkowski space $(\mathbb{R}^{1+d}, \mathfrak{m})$ in the sense that they are *Killing vector fields of* $\mathfrak{m}$*:* $\mathcal{L}_Z \mathfrak{m} = 0$ whenever Z is any of the vector fields mentioned above and $\mathcal{L}$ denotes Lie differentiation. It is in this way that one can argue that the vector field method identifies the geometry of Minkowski space with the geometry of the equation (2.2.1).

Analogizing to the Vlasov model problem of Section 1.2, the natural candidates to provide temporal decay for solutions of (2.2.1) are the boosts L^i for $i = 1, \ldots, d$ because of their t-weight. We note, however, that the crucial step in the proof of Lemma 1.2.9 subtly used that W_i were tangent to constant t-hypersurfaces. This is not the case for the Lorentz boosts. Instead, we are led to consider Sobolev inequalities on hypersurfaces to which L^i are tangent, namely, the hyperboloids Σ_τ of the previous section. This is a fundamental reason as to why we use the hyperboloidal foliations in our analysis. We also emphasize that we have identified the vector fields mentioned in Point **I** in the Prologue of Chapter 1.

The vector field method for the wave and Klein–Gordon equation (2.2.1) has its roots in the general idea of "multiply the equation by $\partial_t \phi$ and integrate by parts". Indeed, by doing just that, one derives the well known conservation of energy identity

$$\int\limits_{\{t\}\times\mathbb{R}^d} |\partial_t \phi|^2 + |\nabla_x \phi|^2 + M^2|\phi|^2 \, \mathrm{d}x = \int\limits_{\mathbb{R}^d} |g|^2 + |\nabla_x f|^2 + M^2|f|^2 \, \mathrm{d}x. \tag{2.2.3}$$

The following presentation of the vector field method formalizes and generalizes this idea in a way that can be used to derive a family of a priori estimates, of which (2.2.3) is a member.

Let φ be any scalar function (in practice φ will be a solution of the equation (2.2.1)). We define the *energy-momentum tensor associated to* φ to be the following symmetric type $\binom{0}{2}$-tensorfield:

$$Q_{\alpha\beta}[\varphi] \overset{\text{def}}{=} \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2}\mathfrak{m}_{\alpha\beta}(\mathfrak{m}^{-1})^{\mu\nu}\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2}\mathfrak{m}_{\alpha\beta} M^2 \varphi^2. \tag{2.2.4}$$

Given φ and a "multiplier" vector field X, we define the corresponding X-*energy current* to be the vector field

$$^{(X)}\mathcal{J}^\alpha[\varphi] \overset{\text{def}}{=} (\mathfrak{m}^{-1})^{\alpha\beta} Q_{\beta\gamma}[\varphi] X^\gamma. \tag{2.2.5}$$

The dominant energy condition is the following well-known result: $Q[\varphi](X,Y)$ is a positive definite quadratic form in $\partial\varphi$ and φ whenever X and Y are both future-directed (i.e. $X^0, Y^0 > 0$) and timelike. In the case that X is timelike and Y is null, then $Q[\varphi](X,Y)$ is a positive semi-definite quadratic form. These properties are what allow one to construct coercive energies and fluxes for wave equation solutions. For example, we note that

$$Q[\varphi](\partial_t, \partial_t) = \frac{1}{2}|\partial_t \varphi|^2 + \frac{1}{2}|\nabla_x \varphi|^2 + M^2|\varphi|^2.$$

This is precisely the energy density that appears in the standard conservation of energy identity (2.2.3). The following lemma makes this precise when $X = \partial_t$ and Y is causal (i.e. null or timelike).

Lemma 2.2.1 (Dominant energy condition)**.** *Let Y be a future oriented causal vector field, that is,* $\mathfrak{m}(Y,Y) \leq 0$ *and* $Y^0 > 0$*. Then*

$$\mathfrak{m}\left({}^{(\partial_t)}\mathcal{J}[\phi], Y\right) \geq 0. \tag{2.2.6}$$

Proof. We note that Y being causal implies

$$(Y^0)^2 \geq \sum_{i=1}^{d} (Y^i)^2.$$

Then we compute by completing the square

$$\begin{aligned}
\mathfrak{m}\left({}^{(\partial_t)}\mathcal{J}[\varphi], Y\right) &= \mathfrak{m}_{\alpha\beta}{}^{(\partial_t)}\mathcal{J}[\varphi]^{\alpha} Y^{\beta} = \mathfrak{m}_{\alpha\beta}(\mathfrak{m}^{-1})^{\alpha\mu} Q[\varphi]_{\mu\nu}(\partial_t)^{\nu} Y^{\beta} = Q[\varphi]_{\mu\nu}(\partial_t)^{\nu} Y^{\mu} \\
&= \partial_t\varphi Y\varphi - \frac{1}{2}\mathfrak{m}(\partial_t, Y)\mathfrak{m}^{-1}(\mathrm{d}\varphi, \mathrm{d}\varphi) - \frac{1}{2}\mathfrak{m}(\partial_t, Y) M^2\varphi^2 \\
&= \frac{1}{2} Y^0(\partial_t\varphi)^2 + \sum_{i=1}^{d} Y^i \partial_t\varphi\partial_i\varphi + \frac{1}{2} Y^0(\partial_i\varphi)^2 + \frac{1}{2} Y^0 M^2\varphi^2 \\
&= \frac{1}{2} Y^0(\partial_t\varphi)^2 + \sum_{i=1}^{d}\left(\frac{\sqrt{Y^0}}{\sqrt{2}}\partial_i\varphi + \frac{Y^i}{\sqrt{2Y^0}}\partial_t\varphi\right)^2 - \frac{(Y^i)^2}{2Y^0}(\partial_t\varphi)^2 + \frac{1}{2} Y^0 M^2\varphi^2 \\
&\geq \frac{1}{2} Y^0(\partial_t\varphi)^2 + \sum_{i=1}^{d}\left(\frac{\sqrt{Y^0}}{\sqrt{2}}\partial_i\varphi + \frac{Y^i}{\sqrt{2Y^0}}\partial_t\varphi\right)^2 - \frac{Y^0}{2}(\partial_t\varphi)^2 + \frac{1}{2} Y^0 M^2\varphi^2 \\
&= \sum_{i=1}^{d}\left(\frac{\sqrt{Y^0}}{\sqrt{2}}\partial_i\varphi + \frac{Y^i}{\sqrt{2Y^0}}\partial_t\varphi\right)^2 + \frac{1}{2} Y^0 M^2\varphi^2 \\
&\geq 0.
\end{aligned}$$

□

Motivated by the dominant energy condition, for any spacelike hypersurface $\tilde{\Sigma}$ with future oriented unit normal $\tilde{N}$ (which is timelike by definition), and for any future oriented timelike multiplied vectorfield X, we define the *X-energy of φ along $\tilde{\Sigma}$* to be

$$\left(\int_{\tilde{\Sigma}} Q[\varphi](X, \tilde{N})\ \mathrm{dvol}_{\tilde{\Sigma}}\right)^{1/2}. \tag{2.2.7}$$

For example, for a solution ϕ of (2.2.1), the ∂_t-energy of ϕ along $\{t\}\times\mathbb{R}^d$, namely

$$\int_{\{t\}\times\mathbb{R}^d} Q[\phi](\partial_t,\partial_t)\ \mathrm{d},$$

is the square of the standard conserved energy of (2.2.3).

A straightforward computation yields the following identity, which will form the starting point for our a priori estimates for the wave and Klein–Gordon equations:

$$\mathrm{div}\Big({}^{(X)}\mathcal{J}[\varphi]\Big) = (\Box_{\mathfrak{m}} - M^2)\varphi X\varphi + \frac{1}{2}Q^{\alpha\beta}[\varphi]\mathcal{L}_X\mathfrak{m}_{\alpha\beta}. \tag{2.2.8}$$

We pause at this juncture to make a few remarks. Firstly, the vector field X was called a "multiplier" because the term $(\Box_{\mathfrak{m}} - M^2\varphi)X\varphi$ on the right hand side of (2.2.8) is being multiplied by $X\varphi$. It is in this way that we are generalizing the previous notion of "multiplying the equation by $\partial_t\phi$" when deriving (2.2.3). Secondly, we note that the second term $\frac{1}{2}Q^{\alpha\beta}[\varphi]\mathcal{L}_X\mathfrak{m}_{\alpha\beta}$ *completely vanishes* whenever X is a Killing field of the Minkowski metric $\mathfrak{m}$. From the discussions in the previous paragraph, for the rest of this work we will always use the multiplier vector field X to be the Killing field ∂_t. The a priori estimates will be a consequence of applying the divergence theorem to $\mathrm{div}\Big({}^{(X)}\mathcal{J}[\varphi]\Big)$ over a spacetime domain sandwiched between *spacelike* hypersurfaces (so that the corresponding normals are timelike in order to obtain coercivity as per the dominant energy condition). This generalizes the notion of "integrating by parts" when deriving the standard conservation of energy identity (2.2.3). It also makes precise the discussion of conservation laws of Point **II** in the prologue of Chapter 1. Indeed, if $\mathrm{supp}\{\phi|_{t=0},\partial_t\phi|_{t=0}\}\Subset B(0,1)$, the proof of Proposition 2.2.2 shows that the energy is conserved because estimate (2.2.9) is achieved with equality.

Proposition 2.2.2. *Suppose ϕ solves $\Box\phi - M^2\phi = 0$. Then*

$$\int_{\Sigma_\tau} Q[\phi](\partial_t,\partial_\tau)\ \mathrm{dvol}_\tau \le \int_{\{0\}\times\mathbb{R}^d} Q[\phi](\partial_t,\partial_t)\,\mathrm{d}x. \tag{2.2.9}$$

Remark 2.2.3. This proof is standard and is adapted from notes taken in Willie Wong's Introduction to Dispersive Equations course.

Proof. Define the regions

$$\mathcal{D}^\tau \overset{\text{def}}{=} \{(t,x) \in \mathbb{R}^{1+d} \mid t > 0, t^2 \leq \tau^2 + |x|^2\}, \qquad \mathcal{D}^\tau_\mu \overset{\text{def}}{=} \mathcal{D}^\tau \cap \{(t,x) \in \mathbb{R}^{1+d} \mid |x| < \mu - t\}.$$

Then we analyze the boundary of $\mathcal{D}^\tau_\mu$ as

$$\begin{aligned} R_\mu &\overset{\text{def}}{=} \{(0,x) \in \mathbb{R}^{1+d} \mid |x| < \mu\}; \\ C_{\tau,\mu} &\overset{\text{def}}{=} \{(t,x) \in \mathbb{R}^{1+d} \mid t^2 - |x|^2 \leq \tau^2,\ t = \mu - |x|\}; \\ \Sigma_{\tau,\mu} &\overset{\text{def}}{=} \{(t,x) \in \mathbb{R}^{1+d} \mid t^2 - |x|^2 = \tau^2,\ t < \mu - |x|\}. \end{aligned}$$

The identity (2.2.8) and the divergence theorem imply

$$0 = \int\limits_{\mathcal{D}^\tau_\mu} (\Box\phi - M^2\phi^2)\partial_t\phi \ \mathrm{dvol} = \int\limits_{\mathcal{D}^\tau_\mu} \mathrm{div}\left({}^{(\partial_t)}\mathcal{J}[\phi]\right) \mathrm{dvol} = \int\limits_{\Sigma_{\tau,\mu}+C_{\tau,\mu}+R_\mu} \iota_{(\partial_t)}\mathcal{J}[\phi] \ \mathrm{dvol}.$$

Along $\Sigma_{\tau,\mu}$, the volume form[2] can be factored as $\mathrm{dvol} = \mathrm{d}\tau \wedge \mathrm{dvol}_\tau$ (see (2.1.4) and (2.1.5)), so

$$\int\limits_{\Sigma_{\tau,\mu}} \iota_{(\partial_t)}\mathcal{J}[\phi] \ \mathrm{d}\tau \wedge \mathrm{dvol}_\tau = \int\limits_{\Sigma_{\tau,\mu}} \mathfrak{m}(\partial_\tau, {}^{(\partial_t)}\mathcal{J}[\phi]) \ \mathrm{dvol}_\tau = \int\limits_{\Sigma_{\tau,\mu}} Q[\phi](\partial_\tau, \partial_t) \ \mathrm{dvol}_\tau.$$

On R_μ, we factor by $\mathrm{dvol} = \mathrm{d}t \wedge \mathrm{d}x$ and use the change in orientation to see

$$\int\limits_{R_\mu} \iota_{(\partial_t)}\mathcal{J}[\phi] \ \mathrm{d}t \wedge \mathrm{d}x = -\int\limits_{t=0,\ |x|<\mu} Q(\partial_t, \partial_t) \ \mathrm{d}x.$$

Finally, we notice that the dominated energy condition (2.2.6) implies

$$\int\limits_{C_{\tau,\mu}} \iota_{(\partial_t)}\mathcal{J}[\phi] \ \mathrm{dvol} \geq 0.$$

[2] Note that the Riemannian metric h induced on Σ_τ by the Minkowski metric agrees with the one on $\Sigma_{\tau,\mu}$

Hence we conclude

$$\int\limits_{\Sigma_{\tau,\mu}} Q(\partial_\tau,\partial_t)\ \mathrm{dvol}_\tau \leq \int\limits_{t=0,\ |x|<\mu} Q(\partial_t,\partial_t)\,\mathrm{d}x \leq \int\limits_{\{0\}\times\mathbb{R}^d} Q(\partial_t,\partial_t)\,\mathrm{d}x.$$

Letting $\mu \to \infty$ concludes the proof. □

Motivated by the discussion immediately following Lemma 2.2.1, given a solution ϕ of (2.2.1), we define *the ∂_t-energy of ϕ along the hyperbola Σ_τ* to be

$$\mathcal{E}_\tau[\phi] \overset{\text{def}}{=} \left(\int\limits_{\Sigma_\tau} Q[\phi](\partial_t,\partial_\tau)\ \mathrm{dvol}_\tau \right)^{1/2}. \tag{2.2.10}$$

In this notation estimate (2.2.9) naturally reads as

$$\mathcal{E}_\tau[\phi] \leq \|f\|_{H^1(\mathbb{R}^d)} + \|g\|_{L^2(\mathbb{R}^d)}.$$

Remark 2.2.4. The flexibility of the dominant energy condition allows one to define other kinds of energies, see (2.2.7). For example, let $K \overset{\text{def}}{=} (t^2+|x|^2)\partial_t + 2\sum_{i=1}^d tx^i\partial_{x^i}$. It is straight forward to check that K is a future oriented causal vector field. Define the $\binom{0}{1}$-tensor

$$\tilde{Q}[\varphi](K,\cdot) \overset{\text{def}}{=} Q[\varphi](K,\cdot) + \frac{d-1}{8(d+1)}\{\mathrm{tr}\,\mathcal{L}_K\mathfrak{m}\}\,\mathrm{d}(\phi)^2 - \frac{d-1}{8(d+1)}\mathrm{d}\{\mathrm{tr}\,\mathcal{L}_K\mathfrak{m}\}\,\varphi^2.$$

and its associated the *modified current*

$${}^{(K)}\widetilde{\mathcal{J}}[\varphi] \overset{\text{def}}{=} \tilde{Q}[\varphi](K,\cdot)^\#. \tag{2.2.11}$$

Then a tedious but straightforward calculation implies the following divergence identity for the modified current:

$$\mathrm{div}\left({}^{(K)}\widetilde{\mathcal{J}}[\varphi]\right) = \Box\phi\,(K\varphi + (d-1)t\varphi).$$

Hence, for solutions to (2.2.1) with $M = 0$, the proof of Proposition 2.2.2 can be replicated[3] to show that

$$\int\limits_{\Sigma_\tau} \tilde{Q}[\phi](\partial_t,\partial_\tau)\ \mathrm{dvol}_\tau \leq \int\limits_{\{0\}\times\mathbb{R}^d} Q[\phi](\partial_t,\partial_t)\,\mathrm{d}x,$$

[3]For compactly supported initial data, say.

where the integral on the left hand side of the above inequality is defined as the *K-energy of ϕ along $\tilde{\Sigma}$.*

Because our analysis will use the weighted Lebesgue and Sobolev spaces (see definitions (2.1.10) – (2.1.12)), we set up the following useful lemma.

Lemma 2.2.5. *Suppose $d \geq 3$. Then the following estimates holds with a universal implicit constant, with $\phi_t \overset{def}{=} \partial_t \phi$:*

$$\mathcal{E}_\tau[\phi] \approx \tau^{-1/2} \|\phi\|_{\mathring{\mathcal{W}}^{1,2}_{-1}} + \tau^{1/2} \|\phi_t\|_{\mathcal{L}^2_{-1}} + \tau^{-1/2} \|M\phi\|_{\mathcal{L}^2_1}, \tag{2.2.12}$$

$$\sum_{|\alpha| \leq k} \mathcal{E}_\tau[L^\alpha \phi] \approx \tau^{-1/2} \|\phi\|_{\mathcal{W}^{k+1,2}_{-1}} + \tau^{1/2} \|\phi_t\|_{\mathcal{W}^{k,2}_{-1}} + \tau^{-1/2} \|M\phi\|_{\mathcal{W}^{k,2}_1}. \tag{2.2.13}$$

Proof. The proof can be found in Appendix A.2.2 □

Remark 2.2.6. A feature of (2.2.12) is its anisotropy. Consider momentarily the wave equation case of $M = 0$. The classical energy estimates of wave equations control integrals of $|\partial_t \phi|^2 + |\nabla \phi|^2$ where all components appear on equal footing. Here, however, the transversal (to Σ_τ) derivative $\partial_t \phi$ has a different weight compared to the tangential derivatives $L^i \phi$. Noting that by its definition, ∂_t has *unit-sized* coefficients with expressed relative to the standard coordinates of Minkowski space. The coefficients for L^i (within the light cone $\{t > |x|\}$) have size $\approx t$. Therefore an isotropic analogue would be expected to contain integrals of $\frac{1}{t^2}(L^i \phi)^2$ along with integrals of $\partial_t \phi$. This indicates that an isotropic analogue would contain, instead of the integral given in (2.2.12), the integral

$$\tau^{1/2} \|\phi\|_{\mathring{\mathcal{W}}^{1,2}_{-3}} + \tau^{1/2} \|\phi_t\|_{\mathcal{L}^2_{-1}}.$$

In other words, the integral for $L^i \phi$ in the energy (2.2.12) has a *better* w_τ weight than would be expected from an isotropic energy, such as that controlled by the standard energy estimates.

This improvement reflects the fact that the energy estimate described in this section captures the *peeling properties* of linear waves within the energy integral itself. It is well-known that derivatives *tangential to an out-going light-cone* decay faster along the light-cone, than derivatives transverse to the light-cone. As asymptotically hyperboloids approximate light-cones, we expect the same peeling property to survive. Indeed, the energy inequality (2.2.9) shows that we can capture this in the integral sense.

Proposition 2.2.7. ([Won17b, Proposition 3.2, Remark 5.8]). *Let ϕ solve* (2.2.1). *Then the following estimate holds (with the implicit constant depending only on d)*

$$M\left\|w_\tau^{d/2}\phi\right\|_{L^\infty(\Sigma_\tau)}+\tau\left\|w_\tau^{(d-2)/2}\partial_t\phi\right\|_{L^\infty(\Sigma_\tau)}+\sum_{i=1}^{d}\left\|w_\tau^{(d-2)/2}L^i\phi\right\|_{L^\infty(\Sigma_\tau)} \lesssim \sum_{|\alpha|\leq\lfloor d/2\rfloor+1}\mathcal{E}_\tau[L^\alpha\phi]. \quad (2.2.14)$$

Proof. The proof follows from the global Sobolev pointwise L^2–L^∞ estimate provided by Theorem 2.1.1 and Lemma 2.2.5 by setting $\ell=-1$ for $\partial_t\phi$ and $L^i\phi$ and by setting $\ell=1$ for ϕ. □

Remark 2.2.8. We note that the Klein–Gordon mass term $M\phi$ has *improved* decay over the wave derivatives $\partial_t\phi$ and $L^i\phi$. This is a well-known fact [Kla85a] and is a consequence of the *positive* w_τ-weight in the mass term $M\phi$ of the energy density (see (2.2.12) and (A.2.17)). Note that Proposition 2.2.7 does not provided pointwise decay for ϕ itself in the wave equation case of $M=0$. In this scenario, one can still get decay for ϕ itself by appealing to Hardy's inequality (see Lemma A.2.5) when $d\geq 3$.

Proposition 2.2.9. *Suppose $d\geq 3$. Let ϕ solve the linear wave equation, that is,* (2.2.1) *with $M=0$. Then the following estimate holds (with the implicit constant depending only on d)*

$$\|w_\tau^{(d-2)/2}\phi\|_{L^\infty(\Sigma_\tau)}\lesssim\sum_{|\alpha|\leq\lfloor d/2\rfloor}\mathcal{E}_\tau[L^\alpha\phi]. \quad (2.2.15)$$

Proof. The estimate follows from the global Sobolev pointwise estimate provided by Theorem 2.1.1, Lemma 2.2.5 with $\ell=-1$, by Hardy's inequality (see Lemma A.2.5) to control the term without any derivatives, and by the coercivity afforded by Lemma 2.2.5. □

Remark 2.2.10. Note that estimate (2.2.15) is not sharp. By estimating the fundamental solution of the linear wave equation, we expect ϕ to decay like $t^{(d-1)/2} = w_\tau^{(d-1)/2}$. The sharp decay rate can be obtained by hyperboloidal techniques by using the K-energy associated to ϕ (see Remark 2.2.4 and [Won17b, Proposition 5.6]). *We omit this computation because we do not use the sharp decay rate for ϕ in Chapter 3 nor in 4 when $M = 0$.*

2.2.2 Wave equation, $d = 2, 3, 4$

In this section we apply our results to obtain $\mathcal{L}^r_*$ bounds by $\mathcal{L}^2_*$ integrals that occur as part of the conserved energy for the linear wave equation. As we will see there is often more than one way to obtain interpolated estimates, depending on the number of derivatives one is willing to sacrifice. Rather than attempt to be exhaustive in this section, we will opt for concreteness and list several possible estimates for dimensions $d = 2, 3, 4$, where the choices are more limited. As we will see, the most delicate case is when $d = 2$ because Hardy is not available. For clarity we save those estimates until the end of this subsection.

Throughout we will let u be a smooth function on $\mathbb{R}^{1+d}$, and u_t will denote its time derivative. Motivated by Lemma 2.2.5, we will denote by $\mathfrak{E}_k$ the kth order energy quantity

$$\mathfrak{E}_k(\tau) = \tau^{-1/2}\|u\|_{\mathcal{W}^{k+1,2}_{-1}(\Sigma_\tau)} + \tau^{1/2}\|u_t\|_{\mathcal{W}^{k,2}_{-1}(\Sigma_\tau)}.$$

Proposition 2.2.11 ($d = 3$)**.** *When $r \in [2, 6]$,*

$$\tau^{-1/r}\|u\|_{\mathcal{L}^r_{r/2-2}(\Sigma_\tau)} \lesssim \mathfrak{E}_0(\tau), \tag{2.2.16}$$

$$\tau^{-1/r}\left(\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r/2-2}(\Sigma_\tau)} + \tau\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r/2-2}(\Sigma_\tau)}\right) \lesssim (\mathfrak{E}_k(\tau))^{\frac{6-r}{2r}} \cdot (\mathfrak{E}_{k+1}(\tau))^{\frac{3r-6}{2r}}. \tag{2.2.17}$$

When $r > 6$,

$$\tau^{-1/r}\|u\|_{\mathcal{L}^r_{r/2-2}(\Sigma_\tau)} \lesssim (\mathfrak{E}_0(\tau))^{\frac{r+6}{2r}} \cdot (\mathfrak{E}_1(\tau))^{\frac{r-6}{2r}}, \tag{2.2.18}$$

$$\tau^{-1/r}\left(\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r/2-2}(\Sigma_\tau)} + \tau\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r/2-2}(\Sigma_\tau)}\right) \lesssim (\mathfrak{E}_{k+1}(\tau))^{\frac{r+6}{2r}} \cdot (\mathfrak{E}_{k+2}(\tau))^{\frac{r-6}{2r}}. \tag{2.2.19}$$

For higher derivatives, the latter of the above estimate in $r > 6$ can be replaced by

$$\tau^{-1/r}\left(\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r/2-2}(\Sigma_\tau)} + \tau\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r/2-2}(\Sigma_\tau)}\right) \lesssim (\mathfrak{E}_k(\tau))^{\frac{4}{2r}} \cdot (\mathfrak{E}_{k+1}(\tau))^{\frac{r-2}{2r}} \cdot (\mathfrak{E}_{k+2})^{\frac{r-2}{2r}} \tag{2.2.20}$$

Proof. Estimate (2.2.16) follows by applying ($\mathrm{GNS_{pqr}}$) with $d = 3$, $q = 2$. Indeed, we see

$$\tau^{(1-\theta)/3}\|u\|_{\mathcal{L}^r_{1-\theta r}} \lesssim \left(\|u\|_{\mathcal{L}^2_{-1}}\right)^{\theta} \cdot \left(\|u\|_{\mathring{\mathcal{W}}^{1,2}_{-1}}\right)^{1-\theta},$$

where $\theta \in [0,1]$ is the solution to

$$\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{6} \qquad \Longrightarrow \qquad \theta = \frac{6-r}{2r}, \quad 1-\theta = \frac{3r-6}{2r}.$$

Rearranging using Hardy on the first factor and the definition of the energy we see that (2.2.16) follows. Similarly, if α is a k-tuple with elements drawn from $\{1,2,3\}$ and v is any function we have

$$\tau^{(1-\theta)/3}\|L^\alpha v\|_{\mathcal{L}^r_{1-\theta r}} \lesssim \left(\|L^\alpha v\|_{\mathcal{L}^2_{-1}}\right)^{\theta} \cdot \left(\|L^\alpha v\|_{\mathring{\mathcal{W}}^{1,2}_{-1}}\right)^{1-\theta},$$

with the same θ as before. Replacing $v \mapsto L^i u$ or u_t, and since we can estimate $\|L^\alpha L^i u\|_{\mathcal{L}^2_{-1}}$ by the kth order energy without invoking Hardy, (2.2.17) follows using the definition of their energies with the respective weights.

For larger r, we first appeal to ($\mathrm{GNAWS_{dqr}}$) with $d = 3$, $q = 6$, and

$$1 + \beta q + \alpha q = 1$$
$$-2(1 + \beta\theta r) + 3\alpha = -\frac{1}{2}$$
$$\theta r = \frac{9}{\frac{3}{2} + r - 6}$$

which is solved by

$$-\alpha = \beta = -\frac{\frac{3}{2} + r - 6}{3 + 2r}.$$

This implies

$$\tau^{\frac{r-6}{3r}}\|u\|_{\mathcal{L}^r_{r/2-2}} \lesssim \|u\|^{6/r}_{\mathcal{L}^6_1} \cdot \|u\|^{(r-6)/r}_{\mathcal{W}^{1,3}_{-1/2}}.$$

Applying (2.2.16) and (2.2.17) to the two terms on the right we get

$$\tau^{\frac{r-6}{3r}} \|u\|_{\mathcal{L}^r_{r/2-2}} \lesssim \left(\tau^{1/6}\mathfrak{E}_0(\tau)\right)^{6/r} \cdot \left(\tau^{1/3}\mathfrak{E}_0(\tau)^{1/2}\mathfrak{E}_1(\tau)^{1/2}\right)^{(r-6)/r}$$

and

$$\tau^{\frac{r-6}{3r}} \|u\|_{\mathring{\mathcal{W}}^{k,r}_{r/2-2}} \lesssim \left(\tau^{1/6}\mathfrak{E}_k(\tau)\right)^{6/r} \cdot \left(\tau^{1/3}\mathfrak{E}_k(\tau)^{1/2}\mathfrak{E}_{k+1}(\tau)^{1/2}\right)^{(r-6)/r}.$$

Rearranging this gives (2.2.18) and (2.2.19)

To find the other estimate for $r > 6$ we appeal to the borderline ($\mathrm{GNAWS_{dqr}}$) inequality slightly differently. Using $d = 3$ and $q = 2$ now, with

$$1 + \beta q + \alpha q = -1,$$

$$(1-d)(1+\theta\beta r) + \alpha d = -1/2,$$

$$1/r = \theta/q + (1-\theta)/(r+1^*),$$

we can solve to find

$$\theta = \frac{6}{r(2r-1)}, \qquad \beta = \frac{(-3)(2r-1)}{2(3+2r)}, \qquad \alpha = \frac{2r-9}{2(3+2r)}.$$

Let α be a k-tuple with elements drawn from $\{1,2,3\}$ and v be any function. Then the inequality reads

$$\left(\tau^{1/3}\right)^{\frac{r-2}{r}} \|L^\alpha v\|_{\mathcal{L}^r_{r/2-2}} \lesssim \left(\|L^\alpha v\|_{\mathcal{L}^2_{-1}}\right)^{2/r} \cdot \left(\|L^\alpha v\|_{\mathcal{W}^{1,3}_{-1/2}}\right)^{\frac{r-2}{r}}.$$

Estimating the second factor using (2.2.17) with $r = 3$ and the choice $v = L^i u$ or u_t, we can then rearrange to obtain (2.2.20). □

Proposition 2.2.12 ($d = 4$)**.** *When $r \in [2,4]$,*

$$\tau^{-1/r}\|u\|_{\mathcal{L}^r_{r-3}(\Sigma_\tau)} \lesssim \mathfrak{E}_0(\tau), \tag{2.2.21}$$

$$\tau^{-1/r}\left(\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r-3}} + \tau\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-3}(\Sigma_\tau)}\right) \lesssim \left(\mathfrak{E}_k(\tau)\right)^{\frac{4-r}{r}} \left(\mathfrak{E}_{k+1}(\tau)\right)^{\frac{2r-4}{r}}. \tag{2.2.22}$$

When $r > 4$,

$$\tau^{-1/r}\|u\|_{\mathcal{L}^r_{r-3}(\Sigma_\tau)} \lesssim (\mathfrak{E}_0(\tau))^{2/r} \cdot (\mathfrak{E}_1(\tau))^{\frac{r-2}{r}} \tag{2.2.23}$$

$$\tau^{-1/r}\left(\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r-3}(\Sigma_\tau)} + \tau\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-3}(\Sigma_\tau)}\right) \lesssim (\mathfrak{E}_k(\tau))^{2/r}\,(\mathfrak{E}_{k+2}(\tau))^{\frac{r-2}{r}}, \tag{2.2.24}$$

or

$$\tau^{-1/r}\|u\|_{\mathcal{L}^r_{r-3}(\Sigma_\tau)} \lesssim (\mathfrak{E}_0(\tau))^{4/r}\,(\mathfrak{E}_1(\tau))^{\frac{r-4}{r}}, \tag{2.2.25}$$

$$\tau^{-1/r}\left(\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r-3}(\Sigma_\tau)} + \tau\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-3}(\Sigma_\tau)}\right) \lesssim (\mathfrak{E}_{k+1}(\tau))^{4/r}\,(\mathfrak{E}_{k+2}(\tau))^{\frac{r-4}{r}}. \tag{2.2.26}$$

Proof. The proofs of (2.2.21) and (2.2.22) are the same as (2.2.16) and (2.2.17) except that now $d = 4$ and θ solves

$$\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{4} \qquad \Longrightarrow \qquad \theta = \frac{4-r}{r}, \quad 1-\theta = \frac{2r-4}{r}.$$

To find estimate for $r > 4$ we appeal to the borderline ($\text{GNAWS}_{\text{dqr}}$) inequality. We will first be applying the inequality with

$$d = 4,$$

$$q = 2,$$

$$1 + \beta q + \alpha q = -1,$$

$$(1-d)(1+\theta\beta r) + \alpha d = 1,$$

$$1/r = \theta/q + (1-\theta)/(r+1^*).$$

These equations are solved by

$$\theta = \frac{8}{r(3r-2)}, \qquad \beta = \frac{(-2)(3r-2)}{4+3r}, \qquad \alpha = \frac{3r-8}{4+3r},$$

and so the weight $1 + \theta\beta r + \alpha r = r - 3$.

Let α be a k-tuple with elements drawn from $\{1,2,3,4\}$ and v be any function. Then

$$\left(\tau^{1/4}\right)^{\frac{r-2}{r}}\|L^\alpha v\|_{\mathcal{L}^r_{r-3}} \lesssim \left(\|L^\alpha v\|_{\mathcal{L}^2_{-1}}\right)^{2/r} \cdot \left(\|L^\alpha v\|_{\mathcal{W}^{1,4}_1}\right)^{\frac{r-2}{r}}.$$

This inequality holds for $r > 2$, so in particular for $r > 4$. If $k = 0$ and $v = u$, then the first factor can be estimated by the energy after invoking Hardy. The second factor can by treated with (GNS$_\mathrm{p}$) because $2^* = 4$:

$$\|u\|_{\mathcal{W}^{1,4}_1} = \|u\|_{\mathcal{L}^4_1} + \|u\|_{\mathring{\mathcal{W}}^{1,4}_1} \lesssim \tau^{-1/4}(\|u\|_{\mathring{\mathcal{W}}^{1,2}_{-1}} + \|u\|_{\mathring{\mathcal{W}}^{2,2}_{-1}}).$$

This gives (2.2.23) after applying the definition of the energy. Again, note that if k is arbitrary and $v = L^i u$, then we do *not* have to invoke Hardy to estimate the first factor by the energy $\tau^{1/r}\mathfrak{E}_k^{2/r}$. On the other hand, if $v = u_t$, the first factor is bounded by $\tau^{-1/r}\mathfrak{E}_k^{2/r}$. The second factor in the case of $v \mapsto (L^i u, u_t)$ can again be treated with (GNS$_\mathrm{p}$). Rearranging the inequalities and using the coercivity of their energies with the respective weights gives (2.2.24).

Alternatively, we can also solve with

$$d = 4$$

$$q = 4$$

$$1 + \beta q + \alpha q = 1$$

$$(1-d)(1+\theta\beta r) + \alpha d = 1$$

$$1/r = \theta/q + (1-\theta)/(r+1^*).$$

Let α be a k-tuple now and compute again with (GNAWS$_\mathrm{dqr}$) and (GNS$_\mathrm{p}$)

$$\left(\tau^{1/4}\right)^{\frac{r-4}{r}} \|L^\alpha v\|_{\mathcal{L}^r_{1+\theta\beta r+\alpha r}} \lesssim \left(\|L^\alpha v\|_{\mathcal{L}^4_1}\right)^{4/r} \cdot \left(\|L^\alpha v\|_{\mathcal{W}^{1,4}_1}\right)^{\frac{r-4}{r}}.$$

The prior equations are solved by

$$\theta = \frac{16}{r(3r-8)}, \qquad -\beta = \alpha = \frac{3r-8}{4+3r}$$

and so the weight $1+\theta\beta r+\alpha r = r-3$. We control each factor with (GNS$_\mathrm{p}$) in the two cases $k = 0$, $v = u$ and arbitrary k and $v = (L^i u, u_t)$ as above. This finishes the proof of (2.2.25) and (2.2.26). □

When $d = 2$, Hardy's inequality is generally unavailable for the wave equation energy. So the kth order energy should only be

$$\mathfrak{E}_k(\tau) = \tau^{-1/2} \sum_{j=1}^{k+1} \|u\|_{\mathring{\mathcal{W}}^{j,2}_{-1}(\Sigma_\tau)} + \tau^{1/2} \|u_t\|_{\mathcal{W}^{k,2}_{-1}(\Sigma_\tau)}.$$

So we cannot in general control $\|u\|_{\mathcal{L}^r_*}$; but we can control the first derivatives of u in $\mathcal{L}^r_*$ with suitable weights.

Proposition 2.2.13. *When $r \in [2, \infty)$,*

$$\tau^{-1/r} \left(\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{-1}(\Sigma_\tau)} + \tau \|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{-1}(\Sigma_\tau)} \right) \lesssim (\mathfrak{E}_k(\tau))^{2/r} (\mathfrak{E}_{k+1}(\tau))^{\frac{r-2}{r}}, \tag{2.2.27}$$

Proof. We appeal to the borderline ($\text{GNAWS}_{\text{dqr}}$) inequality with

$$d = 2,$$

$$q = 2,$$

$$1 + \beta q + \alpha q = -1,$$

$$(1 - d)(1 + \theta \beta r) + \alpha d = -1,$$

$$1/r = \theta/2 + (1 - \theta)/(r + 2).$$

These equations are solved by

$$\theta = \frac{4}{r^2}, \qquad \beta = \frac{-r}{2 + r}, \qquad \alpha = \frac{-2}{2 + r},$$

and so the weight $1 + \theta \beta r + \alpha r = -1$. Let α be a k-tuple and let v be an arbitrary function. Then we compute

$$\left(\tau^{1/2}\right)^{\frac{r-2}{r}} \|L^\alpha v\|_{\mathcal{L}^r_{-1}} \lesssim \left(\|L^\alpha v\|_{\mathcal{L}^2_{-1}} \right)^{2/r} \cdot \left(\|L^\alpha v\|_{\mathcal{W}^{1,2}_{-1}} \right)^{\frac{r-2}{r}}.$$

Replacing $v \mapsto L^i u$ or u_t and using the coercivity of their energies with the respective weights concludes the proof. □

2.2.3 Klein–Gordon equation, $d = 2, 3, 4$

The Klein–Gordon energies control additionally a differently weighted L^2 term. Moreover, as we will see below, it is useful to distinguish between the energies of u and u_t (the latter of which also solves the Klein–Gordon equation). We write the kth order energy as

$$\mathfrak{E}_k[v](\tau) = \tau^{-1/2}\|v\|_{\mathcal{W}^{k+1,2}_{-1}(\Sigma_\tau)} + \tau^{1/2}\|v_t\|_{\mathcal{W}^{k,2}_{-1}(\Sigma_\tau)} + \tau^{-1/2}\|v\|_{\mathcal{W}^{k,2}_{1}(\Sigma_\tau)},$$

where v can play the roll of u or u_t. Here we've assumed $M = 1$ for simplicity. Moreover, we assume that $\tau \geq 1$, so that $\|u\|_{\mathcal{L}^2_{-1}} \leq \|u\|_{\mathcal{L}^2_1}$.

Proposition 2.2.14 ($d = 2$)**.** *When $r > 2$, we have*

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{1}(\Sigma_\tau)} \lesssim \mathfrak{E}_k[u](\tau), \tag{2.2.28}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{r-1}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u](\tau))^{2/r} \cdot (\mathfrak{E}_{k+1}[u])^{\frac{r-2}{r}}, \tag{2.2.29}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{-1}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u](\tau))^{2/r} \cdot (\mathfrak{E}_{k+1}[u])^{\frac{r-2}{r}}, \tag{2.2.30}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r-3}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u](\tau))^{2/r} \cdot (\mathfrak{E}_{k+2}[u](\tau))^{\frac{r-2}{r}}. \tag{2.2.31}$$

For the time derivatives the following estimates hold:

$$\tau^{1-3/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{1}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u_t](\tau))^{2/r} \cdot (\mathfrak{E}_{k+1}[u](\tau))^{\frac{r-2}{r}}, \tag{2.2.32}$$

$$\tau^{-1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-1}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u_t](\tau))^{2/r} \cdot (\mathfrak{E}_{k+1}[u_t](\tau))^{\frac{r-2}{r}}, \tag{2.2.33}$$

$$\tau^{1-1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{-1}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u](\tau))^{2/r} \cdot (\mathfrak{E}_{k+1}[u](\tau))^{\frac{r-2}{r}}, \tag{2.2.34}$$

$$\tau^{1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-3}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u](\tau))^{2/r} \cdot (\mathfrak{E}_{k+1}[u_t](\tau))^{\frac{r-2}{r}}. \tag{2.2.35}$$

Proof. Throughout this proof α will be a k-tuple and v will be an arbitrary function. We solve ($\mathrm{GNAWS}_{\mathrm{dqr}}$) for

$$d = 2$$

$$q = 2$$

$$1 + \beta q + \alpha q = \mu$$

$$(1-d)(1+\theta\beta r) + \alpha d = \nu$$

$$1/r = \theta/2 + (1-\theta)/(r+2),$$

where μ, ν can take the values ± 1. Denoting the weight

$$\rho(\mu,\nu) \stackrel{\text{def}}{=} 1 + \theta\beta r + \alpha r,$$

the borderline inequality yields

$$\left(\tau^{1/2}\right)^{\frac{r-2}{r}} \|L^\alpha v\|_{\mathcal{L}^r_{\rho(\mu,\nu)}} \lesssim \left(\|u\|_{\mathcal{L}^2_\mu}\right)^{2/r} \cdot \left(\|u\|_{\mathcal{W}^{1,2}_\nu}\right)^{\frac{r-2}{r}}. \tag{2.2.36}$$

One explicitly computes the weights as

$$\rho(1,-1) = 1, \qquad \rho(1,1) = r-1, \qquad \rho(-1,-1) = -1, \qquad \rho(-1,1) = r-3.$$

Replacing $v \mapsto (u, u_t)$ in (2.2.36) and using the definition of the energies with their respective weights with $\mu = 1, \nu = -1$ proves (2.2.28), (2.2.32). When $\mu = \nu = 1$, this proves (2.2.29), and (2.2.33). On the other hand, replacing $v \mapsto (L^i u, u_t)$ in (2.2.36) and using the definition of the energies with their respective weights with $\mu = \nu = -1$ shows (2.2.30), (2.2.34). Finally, using $\mu = -1, \nu = 1$ proves (2.2.31), and (2.2.35).

□

Remark 2.2.15. We note that (2.2.30) and (2.2.34) are identical to the estimates (2.2.27) derived for the wave equation. Indeed, the Klein–Gordon and wave t-energies *both* control

$$\tau^{-1/2} \sum_{j=1}^{k+1} \|u\|_{\mathring{\mathcal{W}}^{j,2}_{-1}} + \tau^{1/2} \|u_t\|_{\mathcal{W}^{1,2}_{-1}}.$$

The takeaway is that the mass term $\tau^{-1/2}\|u\|_{\mathcal{L}^2_1}$ allows for estimates with different weights.

Remark 2.2.16. One can summarize the proof of Proposition 2.2.14 by saying that its estimates correspond to the four endpoint cases of $\mu, \nu = \pm 1$ when applying (GNAWS$_{\mathrm{dqr}}$). Of course, various interpolations of these hold. One can interpolate, for example, equation (2.2.28) with (2.2.30) to see, for any $\theta \in [0,1]$,

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{1-2\theta}} \lesssim (\mathfrak{E}_k[u](\tau))^{2\theta/r} \cdot (\mathfrak{E}_{k+1}[u](\tau))^{1-2\theta/r}.$$

For the sake of brevity and clarity, we leave these straightforward computations to the reader.

Proposition 2.2.17 ($d = 3$)**.** *When $r \in [2,6]$,*

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{1}(\Sigma_\tau)} \lesssim \mathfrak{E}_k[u](\tau), \tag{2.2.37}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{3r/2-2}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u])^{\frac{6-r}{2r}} \cdot (\mathfrak{E}_{k+1}[u])^{\frac{3r-6}{2r}}, \tag{2.2.38}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r/2-2}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u])^{\frac{6-r}{2r}} \cdot (\mathfrak{E}_{k+1}[u])^{\frac{3r-6}{2r}}, \tag{2.2.39}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{2r-5}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u])^{\frac{6-r}{2r}} \cdot (\mathfrak{E}_{k+2}[u])^{\frac{3r-6}{2r}}. \tag{2.2.40}$$

For the time derivatives, the following estimates hold:

$$\tau^{3/2-4/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{1}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u_t](\tau))^{\frac{6-r}{2r}} \cdot (\mathfrak{E}_{k+1}[u](\tau))^{\frac{3r-6}{2r}}, \tag{2.2.41}$$

$$\tau^{-1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{3r/2-2}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u_t](\tau))^{\frac{6-r}{2r}} \cdot (\mathfrak{E}_{k+1}[u_t](\tau))^{\frac{3r-6}{2r}}, \tag{2.2.42}$$

$$\tau^{1-1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r/2-2}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u](\tau))^{\frac{6-r}{2r}} \cdot (\mathfrak{E}_{k+1}[u](\tau))^{\frac{3r-6}{2r}}, \tag{2.2.43}$$

$$\tau^{-1/2+2/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{2r-5}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u](\tau))^{\frac{6-r}{2r}} \cdot (\mathfrak{E}_{k+1}[u_t](\tau))^{\frac{3r-6}{2r}}. \tag{2.2.44}$$

When $r > 6$, the following estimates hold:

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{r-1}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot (\mathcal{E}_{k+1}[u])^{\frac{r-2}{r}}, \tag{2.2.45}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{3r/2-2}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.46}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{r/2}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{\frac{r+2}{2r}} \cdot (\mathcal{E}_{k+1}[u](\tau))^{\frac{r-2}{2r}}, \tag{2.2.47}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{r-1}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{\frac{r+2}{2r}} \cdot (\mathcal{E}_{k+2}[u])^{\frac{r-2}{2r}}, \tag{2.2.48}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r-3}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot (\mathcal{E}_{k+2}[u](\tau))^{\frac{r-2}{r}}, \tag{2.2.49}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{3r/2-4}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+2}[u](\tau)^{1/2} \cdot \mathcal{E}_{k+3}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.50}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r/2-2}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.51}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r-3}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u](\tau)^{1/2} \cdot \mathcal{E}_{k+3}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}. \tag{2.2.52}$$

For the time derivatives, we have:

$$\tau^{1/2-2/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{r-1}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u_t](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u_t](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.53}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{3r/2-2}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u_t](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u_t](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u_t](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.54}$$

$$\tau^{1-3/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r/2}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u_t](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.55}$$

$$\tau^{1/2-2/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-1}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u_t](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u_t](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.56}$$

$$\tau^{1/2}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-3}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u_t](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.57}$$

$$\tau^{1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{3r/2-4}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u_t](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u_t](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.58}$$

$$\tau^{1-1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r/2-2}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \tag{2.2.59}$$

$$\tau^{1/2}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-3}(\Sigma_\tau)} \lesssim (\mathcal{E}_k[u](\tau))^{2/r} \cdot \left(\mathcal{E}_{k+1}[u_t](\tau)^{1/2} \cdot \mathcal{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}. \tag{2.2.60}$$

Proof. Throughout this proof α will be a k-tuple and v will be an arbitrary function. For $r \in [2,6]$, we can solve ($\mathrm{GNAWS_{pqr}}$) with

$$d = 3$$

$$q = 2$$

$$p = 2$$

$$1 + \beta q = \mu$$

$$1 - p + \alpha p = \nu$$

$$1/r = \theta/q + (1-\theta)/p^*,$$

where μ, ν can again take the values ± 1. Denoting the weight

$$\rho_{pqr}(\mu,\nu) \overset{\text{def}}{=} 1 + (\theta\beta + (1-\theta)\cdot\alpha)r$$

the interpolation inequality yields

$$\tau^{1/2-1/r} \|L^\alpha v\|_{\mathcal{L}^r_{\rho_{pqr}(\mu,\nu)}} \lesssim \left(\|L^\alpha v\|_{\mathcal{L}^2_\mu}\right)^{\frac{6-r}{2r}} \cdot \left(\|L^\alpha v\|_{\mathcal{W}^{1,2}_\nu}\right)^{\frac{3r-6}{2r}}. \tag{2.2.61}$$

One explicitly computes the weights as

$$\rho_{pqr}(1,-1) = 1, \quad \rho_{pqr}(1,1) = 3r/2 - 2, \quad \rho_{pqr}(-1,-1) = r/2 - 2, \quad \rho_{pqr}(-1,1) = 2r - 5.$$

We note that we are unable to simply replace $v \mapsto u$ in (2.2.61) and use the definition of the energies with their respective weights with $\mu = 1, \nu = -1$ because the second factor in (2.2.61) is the *inhomogeneous* Sobolev norm. To remedy this, we again use the extra mass term in the energy:

$$\begin{aligned} \|L^\alpha u\|_{\mathcal{W}^{1,2}_{-1}} &= \|L^\alpha u\|_{\mathcal{L}^2_{-1}} + \|L^\alpha u\|_{\mathring{\mathcal{W}}^{1,2}_{-1}} \\ &\le \|L^\alpha u\|_{\mathcal{L}^2_{1}} + \|L^\alpha u\|_{\mathring{\mathcal{W}}^{1,2}_{-1}} \\ &\lesssim \tau^{1/2} \mathfrak{E}_k[u]. \end{aligned}$$

Now we can replace $v \mapsto (u, u_t)$ in (2.2.61) to prove (2.2.37), (2.2.41) (note that this problem did not occur for $v = u_t$). When $\mu = \nu = 1$, $v \mapsto (u, u_t)$ in (2.2.61) also proves (2.2.38), and (2.2.42).

On the other hand, replacing $v \mapsto (L^i u, u_t)$ in (2.2.61) and using the definition of the energies with their respective weights with $\mu = \nu = -1$ shows (2.2.39), (2.2.43). Finally, using $\mu = -1, \nu = 1$ proves (2.2.40), and (2.2.44).

For the estimates when $r > 6$, we appeal to the borderline ($\mathrm{GNAWS_{dqr}}$) inequality with

$$\begin{gathered} d = 3, \\ q = 2, \\ 1 + \beta q + \alpha q = \sigma, \\ (1-d)(1+\theta\beta r) + \alpha d = \rho_{pq3}(\mu,\nu), \\ 1/r = \theta/q + (1-\theta)/(r+1^*), \end{gathered}$$

where σ can take the values ± 1 and $\rho_{pq3}(\mu,\nu)$ is as above. This inequality is valid for $r > 2$ so in particular $r > 6$. Denoting the weight

$$\rho_{d=3}(\sigma,\mu,\nu) \stackrel{\text{def}}{=} 1 + \theta\beta r + \alpha r,$$

the borderline inequality yields

$$\left(\tau^{1/3}\right)^{\frac{r-2}{r}} \|L^\alpha v\|_{\mathcal{L}^r_{\rho_{d=3}(\sigma,\mu,\nu)}} \lesssim \left(\|L^\alpha v\|_{\mathcal{L}^2_\sigma}\right)^{2/r} \cdot \left(\|L^\alpha v\|_{\mathcal{W}^{1,3}_{\rho_{pq3}(\mu,\nu)}}\right)^{\frac{r-2}{r}}. \tag{2.2.62}$$

One explicitly computes the weights

$$\begin{aligned} &\rho_{d=3}(1,1,-1) = r-1, && \rho_{d=3}(-1,1,-1) = r-3, \\ &\rho_{d=3}(1,1,1) = 3r/2-2, && \rho_{d=3}(-1,1,1) = 3r/2-4, \\ &\rho_{d=3}(1,-1,-1) = r/2, && \rho_{d=3}(-1,-1,-1) = r/2-2, \\ &\rho_{d=3}(1,-1,1) = r-1, && \rho_{d=3}(-1,-1,1) = r-3. \end{aligned}$$

Note that even though

$$\rho_{d=3}(1,1,-1) = \rho_{d=3}(1,-1,1),$$
$$\rho_{d=3}(-1,1,-1) = \rho_{d=3}(-1,-1,1),$$

that μ, ν are different implies that we have different estimates. We note that replacing $v \mapsto u$, $L^i u$ whenever $\sigma = 1, -1$ (respectively) is not enough to prove the estimates because the second factor in (2.2.62) is

$$\|L^\alpha v\|_{\mathcal{W}^{1,3}_{\rho_{pq3}(\mu,\nu)}} = \|L^\alpha v\|_{\mathcal{L}^3_{\rho_{pq3}(\mu,\nu)}} + \|L^\alpha v\|_{\mathring{\mathcal{W}}^{1,3}_{\rho_{pq3}(\mu,\nu)}}.$$

Consequently, special care must be taken to analyze the two *different* derivative terms because the left hand sides in (2.2.37) - (2.2.40) are all with respect to the *homogeneous* spaces $\mathring{\mathcal{W}}^{*,r}_{*}$.

Fix $v = u$. When $\mu = 1, -1$ we can estimate

$$\tau^{-1/3}\|L^\alpha u\|_{\mathcal{W}^{1,3}_{1}} \lesssim \mathfrak{E}_{k+1}[u]$$

by using $\mathfrak{E}_k \leq \mathfrak{E}_{k+1}$. Arguing in the same way, when $\mu = \nu = 1$ one finds

$$\tau^{-1/3}\|L^\alpha u\|_{\mathcal{W}^{1,3}_{5/2}} \lesssim \mathfrak{E}_{k+1}[u]^{1/2} \cdot \mathfrak{E}_{k+2}[u]^{1/2}.$$

For $\mu = \nu = -1$, on the other hand, we estimate

$$\tau^{-1/3}\|L^\alpha u\|_{\mathcal{W}^{1,3}_{-1/2}} \leq \tau^{-1/3}\left(\|L^\alpha u\|_{\mathcal{L}^3_{5/2}} + \|L^\alpha u\|_{\mathring{\mathcal{W}}^{1,2}_{-1/2}}\right) \lesssim \mathfrak{E}_k[u]^{1/2} \cdot \mathfrak{E}_{k+1}[u]^{1/2}.$$

The first term was controlled again using (2.2.38). Finally, when $\mu = -1, \nu = -1$ we see

$$\begin{aligned}\tau^{-1/3}\|L^\alpha u\|_{\mathcal{W}^{1,3}_{1}} &= \tau^{-1/3}\left(\|L^\alpha u\|_{\mathcal{L}^3_{1}} + \|L^\alpha u\|_{\mathring{\mathcal{W}}^{1,3}_{1}}\right) \\ &\lesssim \mathfrak{E}_k[u]^{1/2} \cdot \mathfrak{E}_{k+1}[u]^{1/2} + \mathfrak{E}_k[u]^{1/2} \cdot \mathfrak{E}_{k+2}[u]^{1/2} \\ &\lesssim \mathfrak{E}_k[u]^{1/2} \cdot \mathfrak{E}_{k+2}[u]^{1/2}.\end{aligned}$$

Using these estimates in (2.2.62) with $\sigma = 1$ proves (2.2.45) - (2.2.48) after appealing to the definition of the energy with the respective weights.

Fix now $v = L^i u$. Then, arguing as above with $\mathfrak{E}_k \leq \mathfrak{E}_{k+1}$ for arbitrary k to control $\|L^\alpha L^i u\|_{\mathcal{W}^{1,3}_{\rho_{pq3}(\mu,\nu)}}$, equation (2.2.62) with $\sigma = -1$ and the estimates (2.2.37) - (2.2.40) with the respective choices of $\mu, \nu = \pm 1$ prove (2.2.49) - (2.2.52).

The time derivative estimates are more straight forward, the $\sigma = \pm 1$ cases are treated separately but similarly. The first factor in (2.2.62) is treated by

$$\|L^\alpha u_t\|_{\mathcal{L}^2_1} \leq \tau^{1/2}\mathfrak{E}_k[u_t], \qquad \|L^\alpha u_t\|_{\mathcal{L}^2_{-1}} \leq \tau^{-1/2}\mathfrak{E}_k[u].$$

Simply replacing $v \mapsto u_t$ in (2.2.62) and using (2.2.41) - (2.2.44) to control the second factor $\|L^\alpha u_t\|_{\mathcal{W}^{1,3}_{\rho_{pq3}(\mu,\nu)}}$ with the respective choices of $\mu, \nu = \pm 1$ proves (2.2.53) - (2.2.60) after appealing to the energies with the respective weights. □

Remark 2.2.18. For the estimates when $r > 6$ in the previous proof we made the choice of interpolating $\mathcal{L}^2_*$ with $\mathcal{W}^{1,3}_*$, see (2.2.62). As we saw previously in the wave case, specifically the proof of (2.2.19), we can also obtain estimates interpolating $\mathcal{L}^6_*$ with $\mathcal{W}^{1,3}_*$ instead. For brevity we leave out these cases and various other interpolations.

Proposition 2.2.19 ($d = 4$)**.** *When $r \in [2, 4]$,*

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_1(\Sigma_\tau)} \lesssim \mathfrak{E}_k[u](\tau), \tag{2.2.63}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{2r-3}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u])^{\frac{4-r}{r}} \cdot (\mathfrak{E}_{k+1}[u])^{\frac{2r-4}{r}}, \tag{2.2.64}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r-3}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u])^{\frac{4-r}{r}} \cdot (\mathfrak{E}_{k+1}[u])^{\frac{2r-4}{r}}, \tag{2.2.65}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{3r-7}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u])^{\frac{4-r}{r}} \cdot (\mathfrak{E}_{k+2}[u])^{\frac{2r-4}{r}}. \tag{2.2.66}$$

For the time derivatives, the following estimates hold:

$$\tau^{2-5/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{1}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u_t](\tau))^{\frac{4-r}{r}} \cdot (\mathfrak{E}_{k+1}[u](\tau))^{\frac{2r-4}{r}}, \tag{2.2.67}$$

$$\tau^{-1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{2r-3}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u_t](\tau))^{\frac{4-r}{r}} \cdot (\mathfrak{E}_{k+1}[u_t](\tau))^{\frac{2r-4}{r}}, \tag{2.2.68}$$

$$\tau^{1-1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-3}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u](\tau))^{\frac{4-r}{r}} \cdot (\mathfrak{E}_{k+1}[u](\tau))^{\frac{2r-4}{r}}, \tag{2.2.69}$$

$$\tau^{-1+3/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{3r-7}(\Sigma_\tau)} \lesssim (\mathfrak{E}_k[u](\tau))^{\frac{4-r}{r}} \cdot (\mathfrak{E}_{k+1}[u_t](\tau))^{\frac{2r-4}{r}}. \tag{2.2.70}$$

When $r > 4$, the following estimates hold:

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{r-1}(\Sigma_\tau)} \lesssim \begin{cases} (\mathfrak{E}_k[u](\tau))^{2/r} \cdot (\mathfrak{E}_{k+1}[u])^{\frac{r-2}{r}}, \\ (\mathfrak{E}_k[u](\tau))^{\frac{r+2}{2r}} \cdot (\mathfrak{E}_{k+1}[u](\tau))^{\frac{r-2}{2r}}, \end{cases} \tag{2.2.71}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k,r}_{2r-3}(\Sigma_\tau)} \lesssim \begin{cases} (\mathfrak{E}_k[u](\tau))^{2/r} \cdot \left(\mathfrak{E}_{k+1}[u](\tau)^{1/2} \cdot \mathfrak{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \\ (\mathfrak{E}_k[u](\tau))^{\frac{r+2}{2r}} \cdot (\mathfrak{E}_{k+2}[u])^{\frac{r-2}{2r}}, \end{cases} \tag{2.2.72}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{r-3}(\Sigma_\tau)} \lesssim \begin{cases} (\mathfrak{E}_k[u](\tau))^{2/r} \cdot (\mathfrak{E}_{k+2}[u](\tau))^{\frac{r-2}{r}}, \\ (\mathfrak{E}_k[u](\tau))^{2/r} \cdot \left(\mathfrak{E}_{k+1}[u](\tau)^{1/2} \cdot \mathfrak{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \end{cases} \tag{2.2.73}$$

$$\tau^{-1/r}\|u\|_{\mathring{\mathcal{W}}^{k+1,r}_{2r-5}(\Sigma_\tau)} \lesssim \begin{cases} (\mathfrak{E}_k[u](\tau))^{2/r} \cdot \left(\mathfrak{E}_{k+2}[u](\tau)^{1/2} \cdot \mathfrak{E}_{k+3}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \\ (\mathfrak{E}_k[u](\tau))^{2/r} \cdot \left(\mathfrak{E}_{k+1}[u](\tau)^{1/2} \cdot \mathfrak{E}_{k+3}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}. \end{cases} \tag{2.2.74}$$

For the time derivatives, we have:

$$\tau^{1-3/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-1}(\Sigma_\tau)} \lesssim \begin{cases} (\mathfrak{E}_k[u_t](\tau))^{2/r}\cdot\left(\mathfrak{E}_{k+1}[u_t](\tau)^{1/2}\cdot\mathfrak{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \\ (\mathfrak{E}_k[u_t](\tau))^{2/r}\cdot\left(\mathfrak{E}_{k+1}[u](\tau)^{1/2}\cdot\mathfrak{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \end{cases} \tag{2.2.75}$$

$$\tau^{-1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{2r-3}(\Sigma_\tau)} \lesssim \begin{cases} (\mathfrak{E}_k[u_t](\tau))^{2/r}\cdot\left(\mathfrak{E}_{k+1}[u_t](\tau)^{1/2}\cdot\mathfrak{E}_{k+2}[u_t](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \\ (\mathfrak{E}_k[u_t](\tau))^{2/r}\cdot\left(\mathfrak{E}_{k+1}[u](\tau)^{1/2}\cdot\mathfrak{E}_{k+2}[u_t](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \end{cases} \tag{2.2.76}$$

$$\tau^{1-1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{r-3}(\Sigma_\tau)} \lesssim \begin{cases} (\mathfrak{E}_k[u](\tau))^{2/r}\cdot\left(\mathfrak{E}_{k+1}[u_t](\tau)^{1/2}\cdot\mathfrak{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \\ (\mathfrak{E}_k[u](\tau))^{2/r}\cdot\left(\mathfrak{E}_{k+1}[u](\tau)^{1/2}\cdot\mathfrak{E}_{k+2}[u](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \end{cases} \tag{2.2.77}$$

$$\tau^{1/r}\|u_t\|_{\mathring{\mathcal{W}}^{k,r}_{2r-5}(\Sigma_\tau)} \lesssim \begin{cases} (\mathfrak{E}_k[u](\tau))^{2/r}\cdot\left(\mathfrak{E}_{k+1}[u_t](\tau)^{1/2}\cdot\mathfrak{E}_{k+2}[u_t](\tau)^{1/2}\right)^{\frac{r-2}{r}}, \\ (\mathfrak{E}_k[u](\tau))^{2/r}\cdot\left(\mathfrak{E}_{k+1}[u](\tau)^{1/2}\cdot\mathfrak{E}_{k+2}[u_t](\tau)^{1/2}\right)^{\frac{r-2}{r}}. \end{cases} \tag{2.2.78}$$

Proof. The proofs of these estimates are treated in the same way as the proof of Proposition 2.2.17, so we merely highlight the differences. For estimates (2.2.63) - (2.2.70) we solve ($\mathrm{GNAWS_{pqr}}$) with

$$d = 4$$

$$q = 2$$

$$p = 2$$

$$1 + \beta q = \mu$$

$$1 - p + \alpha p = \nu$$

$$1/r = \theta/q + (1-\theta)/p^*,$$

where μ, ν can again take the values ± 1. Denoting the weight

$$\rho_{pqr}(\mu,\nu) \stackrel{\text{def}}{=} 1 + (\theta\beta + (1-\theta)\cdot\alpha)r,$$

the interpolation inequality yields

$$\tau^{1/2-1/r}\|L^\alpha v\|_{\mathcal{L}^r_{\rho_{pqr}(\mu,\nu)}} \lesssim \left(\|L^\alpha v\|_{\mathcal{L}^2_\mu}\right)^{\frac{4-r}{r}}\cdot\left(\|L^\alpha v\|_{\mathcal{W}^{1,2}_\nu}\right)^{\frac{2r-4}{r}}. \tag{2.2.79}$$

One explicitly computes the weights as

$$\rho_{pqr}(1,-1) = 1, \quad \rho_{pqr}(1,1) = 2r-3, \quad \rho_{pqr}(-1,-1) = r-3, \quad \rho_{pqr}(-1,1) = 3r-7.$$

Replacing $\mu, \nu = \pm 1$ and $v \mapsto (u, u_t)$ or $(L^i u, u_t)$ in (2.2.79) then proves (2.2.63) - (2.2.70) by following the same analysis as in the proof of Proposition 2.2.17.

For the estimates when $r > 4$, we appeal to the borderline ($\text{GNAWS}_{\text{dqr}}$) inequality with

$$\begin{gathered} d = 4, \\ q = 2, \\ 1 + \beta q + \alpha q = \sigma, \\ (1-d)(1+\theta\beta r) + \alpha d = \rho_{pq4}(\mu,\nu), \\ 1/r = \theta/q + (1-\theta)/(r+1^*), \end{gathered}$$

where σ can take the values ± 1 and $\rho_{pq4}(\mu,\nu)$ is as above. Denoting the weight

$$\rho_{d=4}(\sigma,\mu,\nu) \overset{\text{def}}{=} 1 + \theta\beta r + \alpha r,$$

the borderline inequality yields

$$\left(\tau^{1/4}\right)^{\frac{r-2}{r}} \|L^\alpha v\|_{\mathcal{L}^r_{\rho_{d=4}(\sigma,\mu,\nu)}} \lesssim \left(\|L^\alpha v\|_{\mathcal{L}^2_\sigma}\right)^{2/r} \cdot \left(\|L^\alpha v\|_{\mathcal{W}^{1,4}_{\rho_{pq4}(\mu,\nu)}}\right)^{\frac{r-2}{r}}. \tag{2.2.80}$$

This inequality is valid for $r > 2$ so in particular $r > 4$. One explicitly computes the weights

$$\begin{aligned} &\rho_{d=4}(1,1,-1) = r-1, && \rho_{d=4}(-1,1,-1) = r-3, \\ &\rho_{d=4}(1,1,1) = 2r-3, && \rho_{d=4}(-1,1,1) = 2r-5, \\ &\rho_{d=4}(1,-1,-1) = r-1, && \rho_{d=4}(-1,-1,-1) = r-3, \\ &\rho_{d=4}(1,-1,1) = 2r-3, && \rho_{d=4}(-1,-1,1) = 2r-5. \end{aligned}$$

Replacing $\sigma, \mu, \nu = \pm 1$ and $v \mapsto (u, u_t)$ or $(L^i u, u_t)$ in (2.2.79) then proves (2.2.71) - (2.2.78) by following the same analysis as in the proof of Proposition 2.2.17. □

Remark 2.2.20. We note that even though the estimates for $r > 4$ in Proposition 2.2.19 had almost the same proofs as the ones for $r > 6$ in Proposition 2.2.17, there is a notable difference between the two: there are only *four* distinct weights for $\rho_{d=4}(\pm 1, \pm 1, \pm 1)$ while there are *six* distinct weights for $\rho_{d=3}(\pm 1, \pm 1, \pm 1)$. The reason for this is that we controlled the second factor of (2.2.61) using the non-borderline estimates derived from ($\mathrm{GNAWS_{pqr}}$) with $3 \in [2, 6]$. On the other hand, the second factor of (2.2.79) was estimated with the *end point* $4 \in [2, 4]$.

CHAPTER 3

TOTALLY GEODESIC WAVE MAPS

3.1 Introduction

Geodesics are a central object of study in both Riemannian and Lorentzian geometry. In the former, they are the curves representing the shortest paths between nearby points. In the latter, timelike geodesics describe the motion of a free falling test particle. It is therefore not surprising that functions between manifolds that preserve geodesics have received extensive attention by mathematicians. More precisely, a map $f : N \to M$ between pseudo-Riemannian manifolds is said to be *totally geodesic* if it maps geodesics to geodesics.

Particular attention has been payed to totally geodesic maps in the elliptic setting because they are automatically *harmonic maps*. A map $\phi : N \to M$ between Riemannian manifolds is said to be harmonic if it is a critical point of the energy functional

$$\mathcal{S}[\phi] \stackrel{\text{def}}{=} \frac{1}{2} \int_N \langle \mathrm{d}\phi, \mathrm{d}\phi \rangle_{T^*N \otimes \phi^{-1}TM} \, \mathrm{dvol}_h. \tag{3.1.1}$$

In local coordinates on the target (M, g), the Euler-Lagrange equations (ELE) take the form

$$\Delta_h \phi^i + \Gamma_{jk}^i(\phi) \langle \mathrm{d}\phi^j, \mathrm{d}\phi^k \rangle_h = 0. \tag{3.1.2}$$

Here Δ_h is the Laplace-Beltrami operator on (N, h) and $\Gamma_{jk}^i(\phi)$ are the Christoffel symbols of M evaluated along the image of ϕ. Harmonic maps simultaneously generalize geodesics and harmonic functions, as can be seen directly from the equation (3.1.2) and setting $N = \mathbb{R}$ or $M = \mathbb{R}$, respectively. The history of totally geodesic maps and harmonic maps has been intertwined since the foundational paper of Eells and Sampson [ES64]. There the authors gave restrictions on the curvatures of M and N that imply the existence of harmonic maps, as well as sufficient conditions for harmonic maps to be totally geodesic.

Our work on totally geodesic maps in this chapter serves to expand the literature to the Lorentzian regime by setting $(N,h) = (\mathbb{R}^{1+3}, \mathfrak{m})$. In this case (3.1.2) becomes a hyperbolic equation, and analyzing the solution ϕ amounts to an initial value problem. In this setting we say a solution to the ELE is a *wave map* if it solves

$$\Box_{\mathfrak{m}} \phi^i + \Gamma^i_{jk}(\phi)\langle \mathrm{d}\phi^j, \mathrm{d}\phi^k \rangle_{\mathfrak{m}} = 0. \tag{3.1.3}$$

The theory of wave maps has a rich history, for a general review see [SS98, Kri07]. Our results apply to the physically relevant cases where the target $M^n = \mathbb{S}^n$ or $\mathbb{H}^n$. The former case models the nonlinear sigma model in plasma physics [GML60], while the latter has applications in general relativity [CBM96].

The starting point of our discussion is motivated by the Riemannian setting. A well known result of Vilms shows that "if N is complete, then every totally geodesic map $\phi : N \to M$ factors as

$$N \xrightarrow{\Phi_{\mathrm{S}}} B \xrightarrow{\Phi_{\mathrm{I}}} M; \tag{3.1.4}$$

with Φ_{S} a Riemannian submersion and Φ_{I} a Riemannian immersion, both being totally geodesic" [Vil70]. This chapter is concerned with the global stability of certain infinite energy totally geodesic maps from Minkowski space $\mathbb{R}^{1+d}$ with $d \geq 3$ into a spaceform (M^n, g). We consider as our background solutions those mappings that factor as

$$\mathbb{R}^{1+d} \xrightarrow{\varphi_{\mathrm{S}}} \mathbb{R} \xrightarrow{\varphi_{\mathrm{I}}} M; \tag{3.1.5}$$

where, denoting by e the standard Euclidean metric on $\mathbb{R}$, the mapping φ_{S} is a semi-Riemannian submersion[1] to either $(\mathbb{R}, e)$ or $(\mathbb{R}, -e)$, and φ_{I} is a Riemannian immersion from $(\mathbb{R}, e)$ to (M, g). The semi-Riemannian submersion φ_{S} can be classified as *spacelike*

[1] A semi-Riemannian submersion $\varphi : N \to M$ is necessarily an isometry on the horizontal space normal to fibres. See [O'N83, P. 212] for a precise definition.

or *timelike*[2] depending on whether its codomain $\mathbb{R}$ is considered as being equipped with e or $-e$. Theorem 3.1.1 is a rough version of our main results. See Theorem 3.6.1 for the precise statement of **TL** and Theorem 3.7.1 for the precise statement of **SL**.

Theorem 3.1.1 (Rough version)**.** *Fix $d \geq 3$. A totally geodesic map satisfying the factorization* (3.1.5) *is globally nonlinearly stable as a solution to the initial value problem for the wave maps equation under compactly supported smooth perturbations, provided that either*

TL *φ_S is timelike and (M,g) is a negatively-curved spaceform, or*

SL *φ_S is spacelike and (M,g) is a positively-curved spaceform.*

We emphasize the following key point: *as $\varphi_S : \mathbb{R}^{1+d} \to \mathbb{R}$ is an orthogonal projection onto a 1-dimensional subspace, it automatically satisfies the* ***linear wave equation****. Furthermore, the total geodesy of the composed map trivially implies that* ***the image of φ_I is a geodesic in M*****.** This latter point follows from [ES64, Corollary 5A; (21)].

Stability of factored (non-totally geodesic) wave maps of the form

$$\mathbb{R}^{1+d} \xrightarrow{\varphi_W} \mathbb{R} \xrightarrow{\varphi_G} M$$

has been studied by Sideris and Grigoryan [Sid89, Gri10]. In his paper, Sideris was motivated to study the stability of wave maps *localized* to a geodesic to overcome singularity issues discovered in [Sha88], where singular solutions for the nonlinear σ-model $\mathbb{R}^{1+3} \to \mathbb{S}^3$ were constructed whose range contained a hemisphere. Our problem is related to [Sid89, Gri10] in that their background is also the composition of a geodesic φ_G and a solution to the linear wave equation φ_W. Contrastingly, their φ_W is an arbitrary *finite energy* solution to the linear wave equation and hence $\varphi_G \circ \varphi_W$ is not totally geodesic. This provides yet another motivation for our problem where we assume that φ_S is assumed to be a semi-Riemannian submersion and hence has *infinite* total energy. This

[2]Note that by definition, a semi-Riemannian submersion cannot be null. We always equip the real line $\mathbb{R}$, as the domain of φ_I, with $+e$.

introduces considerable difficulties as the finite energy backgrounds of [Sid89, Gri10] *decay* at the expected rate of finite-energy waves, whereas ours are non-decaying.

3.2 Explanation of results

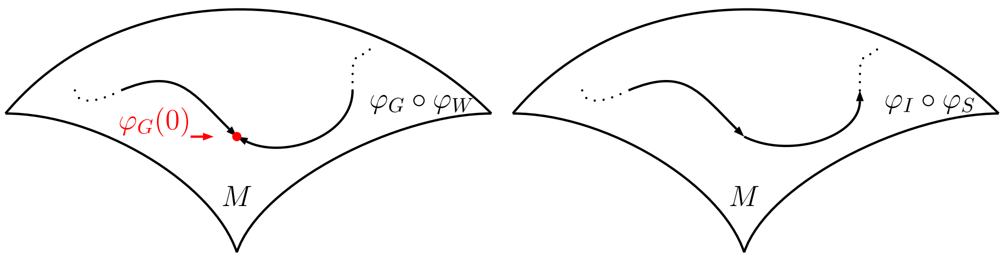
In this section we clarify the geometric set-up for Theorem 3.1.1 and expand on the precise analytical difficulties and conclusions of the result.

In this chapter we adapt the geometric framework of [Sid89, Gri10], where we write the equations of motion for the perturbation in a tubular neighborhood $\mathbb{R}\times\mathcal{N}$ of the geodesic $\varphi_{\mathrm{I}}(\mathbb{R})\subset M$ (here $\mathbb{R}$ parametrizes the geodesic and $\mathcal{N}$ the normal $(n-1)$-directions). The main geometric contribution of this chapter is Proposition 3.4.5, which shows that the equations for the perturbation $\boldsymbol{u}=(u^1,\vec{u})\in\mathbb{R}\times\mathcal{N}$ decouple into a system of wave and Klein–Gordon equations:

$$\begin{cases}\Box u^1 = F^1\boldsymbol{u}\cdot\mathfrak{m}(\mathrm{d}\boldsymbol{u},\mathrm{d}\varphi_{\mathrm{S}})+O(|\boldsymbol{u}|^3+|\partial\boldsymbol{u}|^3),\\ \Box\vec{u}-\vec{M}\vec{u}=\vec{F}\boldsymbol{u}\cdot\mathfrak{m}(\mathrm{d}\boldsymbol{u},\mathrm{d}\varphi_{\mathrm{S}})+O(|\boldsymbol{u}|^3+|\partial\boldsymbol{u}|^3).\end{cases}\tag{3.2.1}$$

Here F^1, $\vec{F}$ are functions of the curvature of (M,g) restricted to the geodesic φ_{I}. The $\vec{M}$ are the masses of $\vec{u}$, and as a consequence of the spaceform assumption on M, Proposition 3.4.5 implies $\vec{M}=\kappa\mathfrak{m}(\mathrm{d}\varphi_{\mathrm{S}},\mathrm{d}\varphi_{\mathrm{S}})$ where κ is the sectional curvature of M. Hence, the assumptions on φ_{S} in Theorem 3.1.1 are there to at minimum guarantee linear stability, i.e. make the Klein–Gordon terms $\vec{u}$ have positive masses.

The computations leading to Proposition 3.4.5 and (3.2.1) hinge on a careful Taylor expansion of the Christoffel symbols Γ about the geodesic $\varphi_{\mathrm{I}}(\mathbb{R})$. This is where our geometric approach differs from that of [Sid89, Gri10]. In their works, the authors need only perform a rough quadratic Taylor expansion because they are able to utilize the decay properties of their background. In our work, we perform a precise *cubic* expansion to capture the lowest order nonlinear structures. Our precise control on these Taylor coefficients reveal weak null-structures that prevent resonant interactions that could lead to finite-time blow up, see (3.4.2) and Lemma 3.4.4. Finally, we remark that the geometry of the target manifold M in [Sid89, Gri10] is arbitrary. Morally, the premise for their stability result is that their background solution converges to the same point in $\varphi_{\mathrm{G}}(0) \in M$ (as a consequence of finite energy!) as one moves in any direction on $\mathbb{R}^{1+d}$ to infinity. As our background is not decaying, moving along generic directions on $\mathbb{R}^{1+d}$ does not imply that the image $\varphi_{\mathrm{I}} \circ \varphi_{\mathrm{S}}$ in M converges to a single point. Our spaceform assumption is then a natural way to ensure some sort of homogeneity of the geometry of M as one moves towards infinity on $\mathbb{R}^{1+d}$ along the mapping $\varphi_{\mathrm{I}} \circ \varphi_{\mathrm{S}}$. See Figure 3.1 below.



(a) The image $\varphi_{\mathrm{G}} \circ \varphi_{\mathrm{W}} \subset M$ (b) The image $\phi_{\mathrm{I}} \circ \phi_{\mathrm{S}} \subset M$

Figure 3.1: The two different background solutions.

Remark 3.2.1. As we will see, for the energy estimates of higher derivatives of $\boldsymbol{u}$ we need first and second order commutations of the equations (3.2.1) with the Lorentz boosts $L^i = t\partial_{x^i} + x^i \partial_t$. Under the spaceform assumption, the functions F and $\vec{F}$ are *constant* and hence vanish when differentiated. In the case that the curvature is *not* constant, these coefficients can grow: $L^i F \approx t(F')$. Using the weak null structures revealed in (3.4.2) and (3.4.9), each order of Taylor expansions introduce an additional Klein–Gordon factor (which has a linear decay rate of $|\vec{u}| \lesssim t^{-d/2}$):

$$F = \sum_{|\alpha| \leq N} \partial^\alpha F(0)(\vec{u})^N + O(|\vec{u}|^{N+1}).$$

As $d \geq 3$, this decay can *overcome* the aforementioned growth. And, consequently, we can easily relax the spaceform assumption to targets (M, g) with the following property: *along* $\varphi_I(\mathbb{R})$, *the metric g agrees with a spaceform up to fourth order*. See also Remark 3.3.1

The main analytic contributions of the present chapter are Theorems 3.6.1 and 3.7.1, which are precise versions of 3.1.1. They provide an open set (in a suitable Sobolev topology) of initial data such that the Cauchy problem for (3.2.1) has a global solution in spatial dimension $d = 3$. For our analysis of the equations of motion we use the physical space vector field method and its related energy estimates.

Remark 3.2.2 (Dimensionality)**.** We restrict the proof of the main theorem and the discussions below to $d = 3$ because stability of quadratic wave-Klein–Gordon systems is a known standard result in dimensions $d \geq 4$. This leaves the case of spatial dimension two open for this problem. Recently Ma has made headway in the two dimensional analysis of wave-Klein–Gordon systems [Ma19]. However, using Ma's terminology, the result of [Ma19] does not apply to the "strongly coupled" nonlinearities $\boldsymbol{u} \cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\varphi_S)$ of (3.2.1).

We note that, for $d = 3$, global existence of coupled wave and Klein–Gordon equations is known, see the monograph by LeFloch and Ma [LM14]. In this chapter we give a short proof of their result using a variant (see [Won17b]) of the the hyperboloidal method developed in [LM14]. We remark that in that same article, Wong proved global existence for (3.1.3) for all $d \geq 2$ where ϕ is a small perturbation of a constant. Our main analytical tool is the vector field method adapted to the hyperbolas Σ_τ introduced in Chapter 2. This allows us to avoid using the purely spatial rotations, and as our most important analytic contribution, to prove stability of (3.2.1) assuming that the initial data is in H^3, see Remark 3.2.4. To the best of our knowledge, the best prior results in $d = 3$ using purely physical space techniques for wave-Klein–Gordon systems was stability with initial data at the level of H^6 [LM14].

Remark 3.2.3. We note that there are technical differences between the systems studied in [LM14] and (3.2.1). LeFloch and Ma considered a *quasilinear* system of wave-Klein–Gordon equations, which introduces additional difficulties. On the other hand, their nonlinearities satisfy the *classical* null condition of Klainerman [Kla84] which allows them to extract *improved* decay from all quadratic nonlinearities. We emphasize that our nonlinearities do not satisfy the classical null condition, and hence we are not able to extract the improved decay present in [LM14].

Remark 3.2.4 (Regularity)**.** To guarantee global existence it suffices that the initial perturbation is sufficiently small in $H^3 \times H^2$; this level of smallness is enough to guarantee C^1 convergence. Note that a standard persistence of regularity argument implies that if initial data is in $H^4 \times H^3$ with *smallness* in $H^3 \times H^2$, this guarantees that the solutions remain small in $H^3 \times H^2$, converges to 0 in C^1, and has bounded C^2 norm globally. As we will see, pushing the regularity down to $H^3 \times H^2$ requires our bootstrap mechanism to allow for *growth* in the top order energies, see Proposition 3.6.6. Roughly speaking, this is because the improved linear decay for Klein–Gordon derivatives $|\partial\vec{u}| \lesssim t^{-3/2}$ is available using only the *third* order energies (at the level of H^4). Instead, by sacrificing a decay factor of $t^{-1/2}$, we can rely on the interpolated Sobolev embeddings of [AW19b] to close the argument at the level of H^3. Had we assumed *smallness* in H^4, our arguments could easily be adapted to prevent this growth, guaranteeing C^2 convergence.

What makes our argument run through is that the spaceform curvature restrictions expose hidden weak null structures that make harmful wave–wave resonant terms from the quadratic and cubic nonlinearities vanish. More precisely, we will show that the *undifferentiated* factor u^1 is missing in $\boldsymbol{u} \cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\varphi_{\mathrm{S}})$ and $|\boldsymbol{u}|^3$ in equations (3.2.1). We also show that the quadratic nonlinearity for the wave solution u^1 is of the form $\vec{u} \cdot \partial\vec{u}$.

There are numerous ramifications of these exposed null conditions. Firstly, as we are unable to use the Morawetz vector field as a multiplier, the available decay rate for u^1 in dimension 3 is $t^{-1/2}$. This means that terms of the form $(u^1)^2$ or $(u^1)^3$ (which are excluded by our exposed null structures) could lead to finite-time blow up. Secondly, it is crucial that *only* $\vec{u}$ appear in the quadratic nonlinearity for u^1 because its expected decay rate is $|\vec{u}| + |\partial\vec{u}| \lesssim t^{-3/2}$, compared to the derivative wave decay $|\partial u^1| \lesssim t^{-1}$. This improved decay for the nonlinearity of u^1 will feed-back into the Klein–Gordon equations when we try to estimate $|\vec{u} \cdot \mathfrak{m}(\mathrm{d}u^1, \mathrm{d}\varphi_{\mathrm{S}})|$, allowing us to close our estimates.

3.3 Geodesic normal coordinates

In this section we set up the geometric tools and notations needed for the rest of the sequel. We first consider the case where M is an arbitrary complete Riemannian manifold and later specialize to the spaceform setting.

We will construct a system of coordinates for a tubular neighborhood of an arbitrary geodesic, in which the restriction of the Christoffel symbols to the geodesic vanish. For a comprehensive treatment of the geometry of geodesic normal coordinates, see the book by Alfred Gray [Gra04]. He analyzes a generalization of geodesic normal coordinates called *Fermi coordinates*. They give a local description of a tubular neighborhood about an embedded submanifold $P \subset M$ of arbitrary codimension.

Consider our complete Riemannian manifold (M^n, g) and let $\gamma : \mathbb{R} \to M$ be a fixed geodesic parametrized by arc-length. Let $\mathcal{V} = \{(\gamma(t), v) \mid t \in \mathbb{R},\ v \in T_{\gamma(t)}M^{\perp}\}$ denote the normal bundle along the geodesic γ. We write $\mathcal{V}_\gamma$ for the fibres above γ and $\mathcal{V}_{\gamma(t_0)}$ when we wish to specify the fibre above a specific point $\gamma(t_0)$. We now construct an explicit local orthonormal frame of a subbundle of $\mathcal{V}$ and use it to define the so called geodesic normal coordinates by the exponential map.

We will parametrize the tubular neighborhood of γ by $\mathbb{R} \times \mathcal{N}$, where

$$\mathcal{N} \stackrel{\text{def}}{=} \{\vec{x} = (x^2, \ldots, x^n) \in \mathbb{R}^{n-1} \mid |\vec{x}| < r_{\mathrm{foc}}(\gamma)\}.$$

Here $r_{\mathrm{foc}}(\gamma)$ is the *focal radius of* γ, which is defined to be the maximal radius such that the normal exponential map, see (3.3.1), is non-critical on the normal disc bundle of γ of radius $r_{\mathrm{foc}}(\gamma)$. In the subsequent analysis of the wave map problem, we can guarantee that $r_{\mathrm{foc}}(\gamma) > 0$ because of the spaceform assumption.

Remark 3.3.1. The generalization of Remark 3.2.1 to targets (M,g) such that the metric agrees with a spaceform up to fourth order along the geodesic $\varphi_{\mathrm{I}}(\mathbb{R})$ should also be accompanied with the following assumptions:

- the focal radius $r_{\mathrm{foc}}(\gamma)$ is bounded away from zero;
- higher derivatives of the metric *in geodesic normal coordinates* are bounded away from infinity.

Denote $e_1 \stackrel{\text{def}}{=} \dot{\gamma}(0)$ and use it to define an orthonormal basis

$$e^{\perp} \stackrel{\text{def}}{=} (e_2, \ldots, e_n).$$

of $\mathcal{V}_{\gamma(0)}$. For arbitrary $x^1 \in \mathbb{R}$, let $(e_1(x^1), e^{\perp}(x^1))$ be defined by parallel transporting $(e_1, e^{\perp})$ along γ and note that $e^{\perp}(x^1)$ is an orthonormal frame for $\mathcal{V}_{\gamma(x^1)}$. Also note that $e_1(x^1) = \dot{\gamma}(x^1)$ by definition.

For $(x^1, \vec{x}) \in \mathbb{R} \times \mathcal{N}$, define $\gamma^{\perp}(x^1; \vec{x}, s)$ as the unique geodesic (with path parameter s) defined by

$$\gamma^{\perp}(x^1; \vec{x}, 0) = \gamma(x^1), \qquad \dot{\gamma}^{\perp}(x^1; \vec{x}, 0) = \sum_{k=2}^{n} x^k e_k(x^1).$$

Remark 3.3.2. We identify the original geodesic γ with $\{\vec{x} \equiv 0\}$ because we have $\gamma^{\perp}(x^1; 0, s) = \gamma^{\perp}(x^1; 0, 0) = \gamma(x^1)$ for any $s \in [0,1]$ by uniqueness of ODEs.

We can now define the normal exponential map

$$\exp^{\perp}_{\gamma(x^1)}(\vec{x}) \stackrel{\text{def}}{=} \gamma^{\perp}(x^1, \vec{x}, 1), \tag{3.3.1}$$

which is a map

$$\exp^{\perp}_{\gamma(x^1)} : \mathcal{V}_{\gamma(x^1)} \to M.$$

This normal exponential map shares many features with the usual one from Riemannian geometry. For example, the inverse function theorem and the following computation show that $\mathcal{N}$ is non-trivial and that $\exp^{\perp}_{\gamma(\cdot)}(\cdot)$ is indeed a smooth immersion from $\mathbb{R}\times\mathcal{N}$ to a a tubular neighborhood, which we denote as $\mathcal{T}$, around $\gamma\subset M$:

$$\begin{aligned} \mathrm{d}\left(\exp^{\perp}_{\gamma(x^1)}\right)_{\{\vec{x}=0\}}(\vec{y}) &= \frac{d}{ds}\Big|_{s=0}\exp^{\perp}_{\gamma(x^1)}(s\vec{y}) = \frac{d}{ds}\Big|_{s=0}\gamma(x^1;s\vec{y},1) \\ &= \frac{d}{ds}\Big|_{s=0}\gamma(x^1;\vec{y},s) \\ &= \vec{y}. \end{aligned} \tag{3.3.2}$$

Remark 3.3.3. The "d" in (3.3.2) denoted the differential of the map in the $\vec{x}$ variables. Since $\exp^{\perp}_{\gamma(\cdot)}(\cdot)$ is technically a map on domain $\mathbb{R}\times\mathcal{N}$, in the future we write d_{x^1} as the differential in the x^1 variable. It is easy to see that, restricted to $\{\vec{x}=0\}$,

$$\mathrm{d}_{x^1}\left(\exp^{\perp}_{\gamma(y^1)}\right)_0(e_1) = e_1(y^1).$$

This allows us to define the geodesic normal coordinates by the preimage of the exponential map

$$\mathcal{T}\xrightarrow{\exp_{\gamma}^{-1}}\mathbb{R}\times\mathcal{N}.$$

More explicitly, if

$$q=\exp_{\gamma(x^1)}(\vec{x})\in\mathcal{T},$$

then q can be written in geodesic normal coordinates by $(x^1,\vec{x}) = (\exp^{\perp})^{-1}_{\gamma}(q)$.

Remark 3.3.4. Here $(\exp^{\perp}_{\gamma})^{-1}$ is the pre-image of the exponential map. Technically, $\exp^{\perp}_{\gamma}$ it is not a bona fide diffeomorphism near self intersections of γ. In the case that γ is an embedded geodesic, then $\exp^{\perp}_{\gamma}$ is a true diffeomorphism. As we are in the perturbative regime for the ensuing analysis of the wave maps equation, our considerations are local so we will ignore any self intersections.

The following lemma sets up the key geometric tools that we need for our analysis. Even though the proof is standard, we include it for the sake of completion:

Lemma 3.3.5. *Let* $\frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$ *be the coordinate vector fields defined by* $(x^1, \vec{x})$*. Let* $y^1 \in \mathbb{R}$ *be arbitrary. Then the following identities hold*

$$\left.\frac{\partial}{\partial x^i}\right|_{\gamma(y^1)} = e_i(y^1) \qquad i \in \{1, \ldots, n\} \tag{3.3.3}$$

$$\left.g_{ij}\right|_{\gamma(y^1)} = \delta_{ij} \qquad i, j \in \{1, \ldots, n\} \tag{3.3.4}$$

$$\left.\partial_k g_{ij}\right|_{\gamma(y^1)} = 0 \qquad i, j, k \in \{1, \ldots, n\} \tag{3.3.5}$$

$$\left.\nabla_{\partial_{x^i}} \partial_{x^j}\right|_{\gamma(y^1)} = 0 \qquad i, j \in \{1, \ldots, n\} \tag{3.3.6}$$

$$\left.\Gamma^k_{ij}\right|_{\gamma(y^1)} = 0 \qquad i, j, k \in \{1, \ldots, n\} \tag{3.3.7}$$

$$\left.\partial_m \Gamma^k_{ij}\right|_{\gamma(y^1)} = \left.\partial_m \langle \nabla_{\partial_{x^i}} \partial_{x^j}, \partial_{x^k} \rangle_g\right|_{\gamma(y^1)} \qquad i, j, k, m \in \{1, \ldots, n\} \tag{3.3.8}$$

$$\left.\partial^2_{mp} \Gamma^k_{ij}\right|_{\gamma(y^1)} = \left.\partial^2_{mp} \langle \nabla_{\partial_{x^i}} \partial_{x^j}, \partial_{x^k} \rangle_g\right|_{\gamma(y^1)} \qquad i, j, k, m, p \in \{1, \ldots, n\} \tag{3.3.9}$$

Proof. The proof of (3.3.3) for $i \in \{2, \ldots, n\}$ follows by definition and (3.3.2)

$$\left.\frac{\partial}{\partial x^i}\right|_{\gamma(y^1)} = \mathrm{d}\Big(\exp_{\gamma(y^1)}\Big)_{\vec{x}=0}\Big(e_i(y^1)\Big) = e_i(y^1).$$

The case of $i = 1$ follows from the discussion in Remark 3.3.3:

$$\left.\frac{\partial}{\partial x^1}\right|_{\gamma(y^1)} = e_1(y^1).$$

Equation (3.3.4) follows immediately because $\{e_i(y^1) \mid i = 1, \ldots, n\}$ were defined by parallel transporting an orthonormal set and because parallel transport is an isometry.

The definition of parallel transport implies

$$\left.\nabla_{\partial_{x^1}} \partial_{x^1}\right|_{\gamma(y^1)} = \left.\nabla_{\partial_{x^1}} \partial_{x^j}\right|_{\gamma(y^1)} = 0$$

for $j \in \{2, \ldots, d\}$. Because we are using coordinate vector fields and ∇ is torsion free,

$$\nabla_{\partial_{x^m}} \partial_{x^k} = \nabla_{\partial_{x^k}} \partial_{x^m}$$

even away from γ. This implies

$$\nabla_{\partial_{x^j}} \partial_{x^1}\Big|_{\gamma(y^1)} = 0.$$

Next we note that any $X \in \mathcal{V}_{\gamma(y^1)}$ is tangent to the curve $\exp_{\gamma(y^1)}(sX)$. Since this curve is a geodesic by definition, we have that $\nabla_X X = 0$. In particular we have that

$$0 = \nabla_{\partial_{x^i}+\partial_{x^j}}(\partial_{x^i} + \partial_{x^j}) = \nabla_{\partial_{x^i}} \partial_{x^j} + \nabla_{\partial_{x^j}} \partial_{x^i}$$

whenever $i, j \in \{2, \dots, n\}$. The torsion condition and setting $s = 0$ proves

$$\nabla_{\partial_{x^i}} \partial_{x^j}\Big|_{\gamma(y^1)} = 0.$$

This concludes the proof of (3.3.6).

Using that the Levi-Civita connection is metric, we see

$$\partial_{x^k} g_{ij} = \partial_{x^k} \langle \partial_{x^i}, \partial_{x^j} \rangle_g = \langle \nabla_{x^k} \partial_{x^i}, \partial_{x^j} \rangle_g + \langle \partial_{x^i}, \nabla_{x^k} \partial_{x^j} \rangle_g.$$

Restricting this computation to γ concludes the proof of (3.3.5) using (3.3.6).

For the Christoffel symbols, recall that they are defined by

$$\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}^m \partial_{x^m}.$$

Taking the inner product with ∂_{x^k}, using the metric structure (3.3.4), and (3.3.6) proves

$$0 = \langle \nabla_{\partial_{x^i}} \partial_{x^j}, \partial_{x^k} \rangle_g \Big|_{\gamma(y^1)} = \Gamma_{ij}^k \Big|_{\gamma(y^1)}.$$

Finally, we compute

$$\partial_m \langle \nabla_{\partial_{x^i}} \partial_{x^j}, \partial_{x^k} \rangle_g = \partial_m (\Gamma_{ij}^l g_{lk}) = \partial_m \Gamma_{ij}^l g_{lk} + \Gamma_{ij}^l \partial_m g_{lk}.$$

This and (3.3.4), (3.3.5) show (3.3.8). Similarly, (3.3.9) follows.

□

3.4 Perturbed system and reduction to wave-Klein–Gordon system

We now return to the wave map equation and describe the precise construction for the perturbation to our totally geodesic background

$$\mathbb{R}^{1+d} \xrightarrow{\varphi_{\mathrm{S}}} \mathbb{R} \xrightarrow{\varphi_{\mathrm{I}}} M.$$

Recall that φ_{S} is a semi-Riemannian submersion, so in accordance with the discussion in Section 3.1 regarding [Vil70], we prescribe φ_{S} to be a *linear* function $\ell : \mathbb{R}^{1+d} \to \mathbb{R}$ satisfying $\mathfrak{m}(\mathrm{d}\ell, \mathrm{d}\ell) = \pm 1$. As φ_{I} is an immersed geodesic in M, we identify it with the zero cross-section about the normal bundle of $\varphi_{\mathrm{I}}(\mathbb{R}) \subset M$ (see Remark 3.3.2):

$$\mathbb{R} \overset{\iota}{\hookrightarrow} \mathbb{R} \times \mathcal{N} \xrightarrow{\exp^{\perp}_{\varphi_{\mathrm{I}}}} M;$$

where the first map is the inclusion and the second map is the restriction $\exp^{\perp}_{\varphi_{\mathrm{I}}}\big|_{\vec{x}=0}$.

Equipping $\mathbb{R} \times \mathcal{N}$ with the pull-back metric, we then look for maps of the form $\phi \overset{\text{def}}{=} \iota \circ \ell + \boldsymbol{u}$ which are solutions to the wave maps equation (3.1.3) on[3]

$$\mathbb{R}^{1+d} \xrightarrow{\iota\circ\ell+\boldsymbol{u}} \mathbb{R} \times \mathcal{N}.$$

Here addition is taken coordinate wise on $\mathbb{R} \times \mathcal{N}$. Consequently, the perturbation of our totally geodesic background takes the form

$$\exp^{\perp}_{\varphi_{\mathrm{I}}}(\iota \circ \ell + \boldsymbol{u}) : \mathbb{R}^{1+d} \longrightarrow M$$

which is also a solution to the wave maps equation. Of course, $\boldsymbol{u} \equiv 0$ corresponds to the background $\varphi_{\mathrm{I}} \circ \varphi_{\mathrm{S}}$. As we consider ℓ fixed, the equations of motion (3.1.3) for ϕ reduce to a Cauchy problem for the perturbation $\boldsymbol{u}$.

Let $(x^1, \ldots, x^n)$ be the geodesic normal coordinates about φ_{I} constructed in section 3.3. In these coordinates $\boldsymbol{u} = (u^1, u^2, \ldots, u^n) = (u^1, \vec{u})$ and hence ϕ takes the form

$$\phi = (\ell + u^1, u^2, \ldots, u^n) = (\ell + u^1, \vec{u}),$$

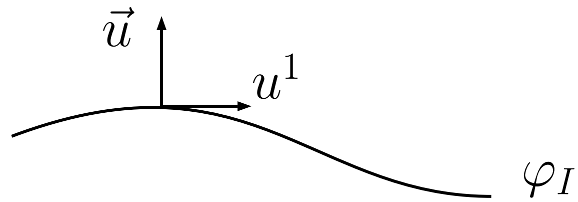


Figure 3.2: The perturbation as a section of the normal bundle about φ_{I}.

see Figure 3.2.

The equations of motion (3.1.3) take the form

$$\begin{aligned} &\Box u^1 + \Gamma^1_{jk}\left(\ell + u^1, \vec{u}\right) \cdot \mathfrak{m}(\mathrm{d}\phi^j, \mathrm{d}\phi^k) = 0, \\ &\Box u^i + \Gamma^i_{jk}\left(\ell + u^1, \vec{u}\right) \cdot \mathfrak{m}(\mathrm{d}\phi^j, \mathrm{d}\phi^k) = 0, \quad i \in \{2, \ldots, n\}. \end{aligned} \tag{3.4.1}$$

We compute

$$\mathfrak{m}(\mathrm{d}\phi^j, \mathrm{d}\phi^k) = \mathfrak{m}(\mathrm{d}\ell, \mathrm{d}\ell)\delta^j_1\delta^k_1 + \mathfrak{m}(\mathrm{d}\ell, \mathrm{d}u^j)\delta^k_1 + \mathfrak{m}(\mathrm{d}\ell, \mathrm{d}u^k)\delta^j_1 + \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}u^k).$$

Taylor expanding Γ about the geodesic $\varphi_{\mathrm{I}} \circ \ell$ we see

$$\Gamma^i_{jk}(\ell + u^1, \vec{u}) = \Gamma^i_{jk}(\ell, \vec{0}) + \sum_{m=1}^{n} \partial_m \Gamma^i_{jk}(\ell, \vec{0}) u^m + O(|\boldsymbol{u}|^2).$$

We pause at this juncture to make some reductions. From (3.3.7) we see that the first term on the right hand side vanishes. Moreover, since $\Gamma^i_{jk}(\ell, \vec{0}) = 0$ for arbitrary ℓ, we see that

$$\underbrace{\partial_1 \cdots \partial_1}_{q \text{ times}} \Gamma^i_{jk}(\ell, \vec{0}) = 0 \tag{3.4.2}$$

for all $i, j, k \in \{1, \ldots, n\}$ and any positive integer q.

Before we expand the Christoffel symbols up to third order, we introduce the following notation: if A is an m-tuple with elements drawn from $\{1, \ldots, n\}$ (namely that $A = (A_1, \ldots, A_m)$ with $A_i \in \{1, \ldots, n\}$), for a scalar function f we denote

$$\partial_A f \stackrel{\text{def}}{=} \partial_{x^{A_1}} \cdots \partial_{x^{A_n}} f.$$

[3] As we will see, initial data for $\boldsymbol{u}$ can be chosen small enough so that the perturbation $\iota \circ \ell + \boldsymbol{u} = (\ell + u^1, \vec{u}) \in \mathbb{R} \times \mathcal{N}$.

By $|A|$ we refer to its length, namely m. Given a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote

$$\boldsymbol{x}^A \stackrel{\text{def}}{=} x^{A_1} \cdots x^{A_m}.$$

We also introduce the shorthand

$$\sigma \stackrel{\text{def}}{=} \mathfrak{m}(\mathrm{d}\ell, \mathrm{d}\ell) \tag{3.4.3}$$

to denote the size of $\mathrm{d}\ell$ as measured by the Minkowski metric. Expanding out the Christoffel symbols up to third order in $\boldsymbol{u}$, we see that the equations can be expressed as

$$\begin{aligned}\Box u^1 + \sum_{m=2}^{n} \partial_m \Gamma^1_{11}(\ell, \vec{0}) u^m \cdot \sigma = \\ -2 \sum_{m=2}^{d} \partial_m \Gamma^1_{j1}(\ell, \vec{0}) u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell) - \sum_{\substack{|A|=2 \\ A \neq (1,1)}} \partial_A \Gamma^1_{11}(\ell, \vec{0}) \boldsymbol{u}^A \cdot \sigma \\ - \sum_{m=2}^{n} \partial_m \Gamma^1_{jk}(\ell, \vec{0}) u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}u^k) - 2 \sum_{\substack{|A|=2 \\ A \neq (1,1)}} \partial_A \Gamma^1_{j1}(\ell, \vec{0}) \boldsymbol{u}^A \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell) \\ - \sum_{\substack{|A|=3 \\ A \neq (1,1,1)}}^{n} \partial_A \Gamma^1_{11}(\ell, \vec{0}) \boldsymbol{u}^A \cdot \sigma + \text{h.o.t.},\end{aligned} \tag{3.4.4}$$

$$\begin{aligned}\Box u^i + \sum_{m=2}^{n} \partial_m \Gamma^i_{11}(\ell, \vec{0}) u^m \cdot \sigma = \\ -2 \sum_{m=2}^{d} \partial_m \Gamma^i_{j1}(\ell, \vec{0}) u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell) - \sum_{\substack{|A|=2 \\ A \neq (1,1)}} \partial_A \Gamma^i_{11}(\ell, \vec{0}) \boldsymbol{u}^A \cdot \sigma \\ - \sum_{m=2}^{n} \partial_m \Gamma^i_{jk}(\ell, \vec{0}) u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}u^k) - 2 \sum_{\substack{|A|=2 \\ A \neq (1,1)}} \partial_A \Gamma^i_{j1}(\ell, \vec{0}) \boldsymbol{u}^A \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell) \\ - \sum_{\substack{|A|=3 \\ A \neq (1,1,1)}} \partial_A \Gamma^i_{11}(\ell, \vec{0}) \boldsymbol{u}^A \cdot \sigma + \text{h.o.t.}\end{aligned} \tag{3.4.5}$$

Remark 3.4.1. To clarify, the sums involving A on the right hand side of (3.4.4) and (3.4.5) are summing over m-tuples $A = (A_1, \ldots, A_m)$ *excluding* the vertex $A_i = 1$ for all i. That is, for example,

$$\sum_{\substack{|A|=2\\A\neq(1,1)}} \partial_A \Gamma^1_{11} \boldsymbol{u}^\alpha \overset{\text{def}}{=} \sum_{\substack{\alpha_1,\alpha_2=1\\(\alpha_1,\alpha_2)\neq(1,1)}}^{n} \partial_{x^{\alpha_1}} \partial_{x^{\alpha_2}} \Gamma^1_{11} u^{\alpha_1} \cdot u^{\alpha}_2.$$

That we are able to do this is of course a consequence of (3.4.2).

Remark 3.4.2. As stated previously, repeated latin indices are implicitly summed over $\{1, \ldots, n\}$ unless otherwise stated. For example,

$$\sum_{m=2}^{n} \partial_m \Gamma^i_{j1} u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell) \overset{\text{def}}{=} \sum_{\substack{m=2\\j=1}}^{n} \partial_m \Gamma^i_{j1} u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell).$$

Remark 3.4.3. In the equations "h.o.t." represents higher order terms of the form

$$\text{h.o.t.} \lesssim C_M(|\boldsymbol{u}|^4 + |\partial \boldsymbol{u}|^4) \cdot f(\boldsymbol{u}, \partial \boldsymbol{u}),$$

where C_M denotes some constant depending on the derivatives of the Christoffel symbols of the target manifold restricted to the geodesic. Here $f : \mathbb{R}^{n(d+2)} \to \mathbb{R}$ is an arbitrary smooth function.

We are able to find explicit formulas for the coefficients of the linear terms:

Lemma 3.4.4. *Let $i, k, m \in \{1, \ldots, n\}$. Then, restricted to the geodesic φ_I, we have*

$$\partial_m \Gamma^i_{k1}\Big|_{\varphi_I} = R_{m1ki}\Big|_{\varphi_I}.$$

Proof. Denote the coordinate vector fields $\partial_{x^i} = X_i$. Then compute

$$\begin{aligned}
\partial_m \langle \nabla_{X_k} X_1, X_i \rangle_g &= \langle \nabla_{X_m} \nabla_{X_k} X_1, X_i \rangle_g + \langle \nabla_{X_k} X_1, \nabla_{X_m} X_i \rangle \\
&= \langle \nabla_{X_m} \nabla_{X_1} X_k, X_i \rangle_g + \langle \nabla_{X_k} X_1, \nabla_{X_m} X_i \rangle_g \\
&= \langle R(X_m, X_1) X_k, X_i \rangle_g + \langle \nabla_{X_1} \nabla_{X_m} X_k, X_i \rangle_g + \langle \nabla_{X_k} X_1, \nabla_{X_m} X_i \rangle_g \\
&= \langle R(X_m, X_1) X_k, X_i \rangle_g + \partial_1 \langle \nabla_{X_m} X_k, X_i \rangle_g \\
&\qquad - \langle \nabla_{X_m} X_k, \nabla_{X_1} X_i \rangle_g + \langle \nabla_{X_k} X_1, \nabla_{X_m} X_i \rangle_g.
\end{aligned}$$

Restricting to φ_{I}, equations (3.3.6) and (3.3.8) yield

$$\partial_m \Gamma^i_{k1}\Big|_{\varphi_{\mathrm{I}}} = R_{m1ki}\Big|_{\varphi_{\mathrm{I}}} + \partial_1 \Gamma^i_{mk}\Big|_{\varphi_{\mathrm{I}}}.$$

The second term on the right hand side vanishes from the discussion immediately before the statement of the lemma. □

From this lemma we immediately see that $\partial_m \Gamma^1_{11}|_{\varphi_{\mathrm{I}}} = R_{m111}|_{\varphi_{\mathrm{I}}} = 0$ from the anti-symmetric property of the Riemann curvature tensor. On the other hand, $\partial_m \Gamma^i_{11}|_{\varphi_{\mathrm{I}}} = R_{m11i}|_{\varphi_{\mathrm{I}}}$, which in general does not vanish. We have then proved the following proposition.

Proposition 3.4.5. *The perturbation equation* (3.4.1) *decouples into the following system of wave and Klein–Gordon equations for the unknowns* $(u^1, \vec{u})$*:*

$$\begin{aligned}\Box u^1 = &\\ &-2\sum_{m=2}^{d} R_{m1j1}(\ell,\vec{0})u^m \cdot \mathfrak{m}(\mathrm{d}u^j,\mathrm{d}\ell) - \sum_{\substack{|A|=2\\ A\neq(1,1)}} \partial_A \Gamma^1_{11}(\ell,\vec{0})\boldsymbol{u}^A \cdot \sigma \\ &-\sum_{m=2}^{n} \partial_m \Gamma^1_{jk}(\ell,\vec{0})u^m \cdot \mathfrak{m}(\mathrm{d}u^j,\mathrm{d}u^k) - 2\sum_{\substack{|A|=2\\ A\neq(1,1)}} \partial_A \Gamma^1_{j1}(\ell,\vec{0})\boldsymbol{u}^A \cdot \mathfrak{m}(\mathrm{d}u^j,\mathrm{d}\ell) \\ &- \sum_{\substack{|A|=3\\ A\neq(1,1,1)}}^{n} \partial_A \Gamma^1_{11}(\ell,\vec{0})\boldsymbol{u}^A \cdot \sigma + h.o.t.,\end{aligned} \tag{3.4.6}$$

$$
\begin{aligned}
\Box u^i + \sum_{m=2}^{n} R_{m11i}(\ell,\vec{0})u^m\sigma = \\
& - 2\sum_{m=2}^{d} R_{m1ji}(\ell,\vec{0})u^m \cdot \mathfrak{m}(\mathrm{d}u^j,\mathrm{d}\ell) - \sum_{\substack{|A|=2\\ A\neq(1,1)}} \partial_A\Gamma^i_{11}(\ell,\vec{0})\boldsymbol{u}^A\cdot\sigma \\
& - \sum_{m=2}^{n} \partial_m\Gamma^i_{jk}(\ell,\vec{0})u^m\cdot\mathfrak{m}(\mathrm{d}u^j,\mathrm{d}u^k) - 2\sum_{\substack{|A|=2\\ A\neq(1,1)}} \partial_A\Gamma^i_{j1}(\ell,\vec{0})\boldsymbol{u}^A\cdot\mathfrak{m}(\mathrm{d}u^j,\mathrm{d}\ell) \\
& - \sum_{\substack{|A|=3\\ A\neq(1,1,1)}} \partial_A\Gamma^i_{11}(\ell,\vec{0})\boldsymbol{u}^A\cdot\sigma + h.o.t.
\end{aligned}
\tag{3.4.7}
$$

3.4.1 Reductions when M is a spaceform

We now suppose that (M,g) is a spaceform with constant sectional curvature $\kappa \neq 0$. In this case the Riemann curvature tensor has the following form:

$$
R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{3.4.8}
$$

This curvature restriction has the following immediate consequence:

Lemma 3.4.6. *Let $m,p \in \{2,\ldots,n\}$ and denote $\bullet$ for any element of $\{1,\ldots,n\}$. Then, restricted to the geodesic φ_I,*

$$
\partial^2_{1m}\Gamma^{\bullet}_{\bullet 1} = \partial^2_{pm}\Gamma^{\bullet}_{11} = \partial^3_{1\bullet\bullet}\Gamma^{\bullet}_{11} = 0. \tag{3.4.9}
$$

Proof. Denoting the coordinate vector fields ∂_{x^i} as X_i, we have already seen in Lemma 3.4.4 that

$$
\begin{aligned}
\partial_m\langle\nabla_{X_k}X_1, X_i\rangle_g = R_{m1ki} + \partial_1\langle\nabla_{X_m}X_k, X_i\rangle_g - \langle\nabla_{X_m}X_k, \nabla_{X_1}X_i\rangle_g \\
+ \langle\nabla_{X_k}X_1, \nabla_{X_m}X_i\rangle_g
\end{aligned}
\tag{3.4.10}
$$

for any $k, i \in \{1, \ldots, n\}$. Taking ∂_1 of both sides shows

$$\begin{aligned}\partial^2_{1m}\langle\nabla_{X_k}X_1, X_i\rangle_g &= \partial_1 R_{m1ki} + \partial^2_{11}\langle\nabla_{X_m}X_k, X_i\rangle_g - \langle\nabla_{X_1}\nabla_{X_m}X_k, \nabla_{X_1}X_i\rangle_g \\ &\quad - \langle\nabla_{X_m}X_k, \nabla_{X_1}\nabla_{X_1}X_i\rangle_g + \langle\nabla_{X_1}\nabla_{X_k}X_1, \nabla_{X_m}X_i\rangle_g + \langle\nabla_{X_k}X_1, \nabla_{X_1}\nabla_{X_m}X_i\rangle_g.\end{aligned}$$

Restricting this identity on the geodesic proves

$$\partial^2_{1m}\Gamma^i_{k1} = \partial_1 R_{m1ki} + \partial^2_{11}\Gamma^i_{mk}$$

using Lemma 3.3.5. The second term vanishes because of (3.4.2). The first term vanishes using the spaceform restriction (3.4.8) and (3.3.5), proving $\partial^2_{1\bullet}\Gamma^{\bullet}_{\bullet 1} = 0$.

We can instead differentiate (3.4.10) by ∂_p and setting $k = 1$ to deduce

$$\begin{aligned}\partial^2_{pm}\langle\nabla_{X_1}X_1, X_i\rangle_g &= \partial_p R_{m11i} + \partial^2_{p1}\langle\nabla_{X_m}X_1, X_i\rangle_g - \langle\nabla_{X_p}\nabla_{X_m}X_1, \nabla_{X_1}X_i\rangle_g \\ &\quad - \langle\nabla_{X_m}X_1, \nabla_{X_p}\nabla_{X_1}X_i\rangle_g + \langle\nabla_{X_p}\nabla_{X_1}X_1, \nabla_{X_m}X_i\rangle_g + \langle\nabla_{X_1}X_1, \nabla_{X_p}\nabla_{X_m}X_i\rangle_g.\end{aligned}$$

Restricting to the geodesic and using Lemma 3.3.5 similarly proves

$$\partial^2_{pm}\Gamma^i_{11} = \partial_p R_{m11i} + \partial^2_{p1}\Gamma^i_{m1}.$$

Again the curvature term vanishes using the spaceform restriction (3.4.8) and (3.3.5), while the second term vanishes using the already proved $\partial^2_{1\bullet}\Gamma^{\bullet}_{\bullet 1} = 0$ and that regular partial derivatives commute.

We have show that, for *arbitrary* ℓ, $\partial^2_{\bullet\bullet}\Gamma^{\bullet}_{11}(\ell, \vec{0}) = 0$. Arguing as in (3.4.2), this proves $\partial^3_{1\bullet\bullet}\Gamma^{\bullet}_{11}(\ell, \vec{0}) = 0$, as desired.

□

This lemma has important ramifications. Firstly, it shows that the only quadratic terms in (3.4.6) and (3.4.7) are of the form $\vec{u} \cdot \mathfrak{m}(\mathrm{d}\vec{u}, \mathrm{d}\ell)$ and $\vec{u} \cdot \mathfrak{m}(\mathrm{d}u^1, \mathrm{d}\ell)$. Secondly, it shows that the *undifferentiated* wave factor u^1 is *missing* from the nonlinearities. This null-structure allows our argument to run because the missing resonant terms such as $(u^1)^2$ or $(u^1)^3$ could potentially blow up in finite time due to the lack of the availability of the Morawetz multiplier.

Lemmas 3.3.5 and 3.4.6, and Proposition 3.4.5 immediately imply that the perturbation equations simplify to

$$\Box u^1 = -2\kappa \sum_{m=2}^{n} u^m \mathfrak{m}(\mathrm{d}u^m, \mathrm{d}\ell) - 2 \sum_{m,p=2}^{n} \partial^2_{mp}\Gamma^1_{j1}(\ell,\vec{0})u^m u^p \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell)$$
$$+ \sum_{m=2}^{n} \partial_m \Gamma^1_{jk}(\ell,\vec{0})u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}u^k) - \sum_{m,p,q=2}^{n} \partial^3_{mpq}\Gamma^1_{11}(\ell,\vec{0})u^m u^p u^q \cdot \sigma + \text{h.o.t.}, \quad (3.4.11)$$

$$\Box u^i - \kappa u^i \cdot \sigma = 2\kappa u^i \cdot \mathfrak{m}(\mathrm{d}u^1, \mathrm{d}\ell) - 2 \sum_{m,p=2}^{n} \partial^2_{mp}\Gamma^i_{j1}(\ell,\vec{0})u^m u^p \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell)$$
$$+ \sum_{m=2}^{n} \partial_m \Gamma^i_{jk}(\ell,\vec{0})u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}u^k) - \sum_{m,p,q=2}^{n} \partial^3_{mpq}\Gamma^i_{11}(\ell,\vec{0})u^m u^p u^q \cdot \sigma + \text{h.o.t.} \quad (3.4.12)$$

3.4.1.1 Negatively curved case

Without loss of generality, in the case of negative sectional curvature we assume $\kappa \equiv -1$. Consequently we demand that the line ℓ be timelike ($\sigma < 0$) in order to make the masses of the Klein–Gordon solutions $\vec{u}$ positive. Without loss of generality, up to a change of coordinates, $\ell \equiv t$. Equations of motions (3.4.11) and (3.4.12) then reduce to

$$\Box u^1 = -2 \sum_{m=2}^{n} u^m \cdot u^m_t - 2 \sum_{m,p=2}^{n} \partial^2_{mp}\Gamma^1_{j1}(\ell,\vec{0})u^m u^p \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell)$$
$$+ \sum_{m=2}^{n} \partial_m \Gamma^1_{jk}(\ell,\vec{0})u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}u^k) - \sum_{m,p,q=2}^{n} \partial^3_{mpq}\Gamma^1_{11}(\ell,\vec{0})u^m u^p u^q + \text{h.o.t.}, \quad (3.4.13)$$

$$\Box u^i - u^i = 2u^i \cdot u^1_t - 2 \sum_{m,p=2}^{n} \partial^2_{mp}\Gamma^i_{j1}(\ell,\vec{0})u^m u^p \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell)$$
$$+ \sum_{m=2}^{n} \partial_m \Gamma^i_{jk}(\ell,\vec{0})u^m \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}u^k) - \sum_{m,p,q=2}^{n} \partial^3_{mpq}\Gamma^i_{11}(\ell,\vec{0})u^m u^p u^q + \text{h.o.t.} \quad (3.4.14)$$

3.4.1.2 Positively curved case

In the case of positive sectional curvature, we assume $\kappa \equiv +1$ and hence, without loss of generality, we can prescribe $\ell \equiv x^1$. This reduces (3.4.11) and (3.4.12) to

$$\begin{aligned}\Box u^1 = 2\sum_{m=2}^{n} u^m \cdot u^m_{x^1} - 2\sum_{m,p=2}^{n} \partial^2_{mp}\Gamma^1_{j1}(\ell,\vec{0})u^m u^p \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell) \\ + \sum_{m=2}^{n} \partial_m \Gamma^1_{jk}(\ell,\vec{0})u^m \cdot \mathfrak{m}(\mathrm{d}u^j,\mathrm{d}u^k) - \sum_{m,p,q=2}^{n} \partial^3_{mpq}\Gamma^1_{11}(\ell,\vec{0})u^m u^p u^q + \text{h.o.t.}, \quad (3.4.15)\end{aligned}$$

$$\begin{aligned}\Box u^i - u^i = -2u^i \cdot u^1_{x^1} - 2\sum_{m,p=2}^{n} \partial^2_{mp}\Gamma^i_{j1}(\ell,\vec{0})u^m u^p \cdot \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}\ell) \\ + \sum_{m=2}^{n} \partial_m \Gamma^i_{jk}(\ell,\vec{0})u^m \cdot \mathfrak{m}(\mathrm{d}u^j,\mathrm{d}u^k) - \sum_{m,p,q=2}^{n} \partial^3_{mpq}\Gamma^i_{11}(\ell,\vec{0})u^m u^p u^q + \text{h.o.t.} \quad (3.4.16)\end{aligned}$$

Remark 3.4.7. Starting now and for the remainder of the chapter we restrict ourselves to the most difficult case of spatial dimension $d = 3$. This is a borderline case in the sense that the linear decay rate for waves t^{-1} barely misses to be integrable. We will overcome this growth by exploiting the weak null-condition present in (3.4.13) – (3.4.16), namely that resonant wave–wave nonlinearities are *not present*. Instead, we see that the strongest nonlinear interactions are of wave-Klein–Gordon type. Our estimates will close by exploiting the *stronger* linear decay rate of $t^{-3/2}$ for the Klein–Gordon equation.

Using the integrable decay rate of $t^{(1-d)/2}$ when $d \geq 4$ for linear waves, it is straight forward to show that our results hold for higher dimensions as well.

3.5 Preliminary L^2 analysis

In this short section we introduce some minor changes in the notation of the energies introduced for the linear wave and Klein–Gordon equation in Chapter 2. We will denote by

$$\mathcal{E}^W_\tau[u^1] \overset{\text{def}}{=} \left(\int\limits_{\Sigma_\tau} \left(\tau^{-1} \sum_{j=1}^{3} |L^j u^1|^2 + \tau(\partial_t u^1)^2 \right) w_\tau^{-1} \, \mathrm{dvol}_{\Sigma_\tau} \right)^{1/2}, \tag{3.5.1}$$

$$\mathcal{E}^{KG}_\tau[u^i] \overset{\text{def}}{=} \left(\int\limits_{\Sigma_\tau} \left(\tau^{-1} \sum_{j=1}^{3} |L^j u^i|^2 + \tau(\partial_t u^i)^2 \right) w_\tau^{-1} + \tau^{-1}|u^i|^2 w_\tau \, \mathrm{dvol}_{\Sigma_\tau} \right)^{1/2} \tag{3.5.2}$$

as the energies of $(u^1, \vec{u})$. We emphasize that the Klein–Gordon energy $\mathcal{E}^{KG}_\tau[u^i]$ is adapted to Klein–Gordon solutions with mass 1, this corresponds to our assumption of $\kappa = \pm 1$ in Subsubsections 3.4.1.1 and 3.4.1.2 (see also Remark 2.2.8). As a consequence of Lemma 2.2.5, these energies are comparable to the following weighted Lebesgue and Sobolev norms of Chapter 2 (see (2.1.10) – (2.1.12)):

$$\mathcal{E}^W_\tau[u^1] \approx \tau^{-1/2} \|u\|_{\mathring{\mathcal{W}}^{1,2}_{-1}} + \tau^{1/2} \|\partial_t u\|_{\mathcal{L}^2_{-1}},$$
$$\mathcal{E}^{KG}_\tau[u^i] \approx \tau^{-1/2} \|u^i\|_{\mathring{\mathcal{W}}^{1,2}_{-1}} + \tau^{1/2} \|\partial_t u^i\|_{\mathcal{L}^2_{-1}} + \tau^{-1/2} \|u^i\|_{\mathcal{L}^2_1}.$$

The proof of Proposition 2.2.2 can be adapted to derive the following fundamental energy estimate

$$\begin{aligned}\mathcal{E}^W_{\tau_1}[u^1]^2 + \sum_{i=1}^{n} \mathcal{E}^{KG}_{\tau_1}[u^i]^2 \lesssim \mathcal{E}^W_{\tau_0}[u^1]^2 + \sum_{i=1}^{n} \mathcal{E}^{KG}_{\tau_0}[u^i]^2 \\ + \int\limits_{\tau_0}^{\tau_1} \int\limits_{\Sigma_\tau} \Box_{\mathfrak{m}} u^1 \partial_t u^1 + \langle \Box_{\mathfrak{m}} \vec{u} - \vec{u}, \partial_t \vec{u} \rangle \, \mathrm{dvol}_{\Sigma_\tau} \, \mathrm{d}\tau, \end{aligned} \tag{3.5.3}$$

where we wrote

$$\langle \vec{\phi}, \vec{\psi} \rangle \overset{\text{def}}{=} \sum_{i=2}^{n} \phi^i \psi^i.$$

To see how (3.5.3) can be derived, define

$$\mathcal{D}^{\tau_2}_{\tau_1} \overset{\text{def}}{=} \{(t,x) \in \mathbb{R}^{1+d} \mid t > 0, \tau_1^2 \le t^2 - |x|^2 \le \tau_2^2\}, \qquad {}_\mu\mathcal{D}^{\tau_2}_{\tau_1} \overset{\text{def}}{=} \mathcal{D}^{\tau_2}_{\tau_1} \cap \{(t,x) \in \mathbb{R}^{1+d} \mid |x| < \mu - t\}.$$

The boundary of ${}_{\mu}\mathcal{D}_{\tau_1}^{\tau_2}$ is the union of

$$\begin{aligned} C_{\mu} &\stackrel{\text{def}}{=} \{(t,x) \in \mathbb{R}^{1+d} \mid \tau_1^2 \le t^2 - |x|^2 \le \tau_2^2,\ t = \mu - |x|\}; \\ \Sigma_{\tau_2,\mu} &\stackrel{\text{def}}{=} \{(t,x) \in \mathbb{R}^{1+d} \mid t^2 - |x|^2 = \tau_2^2, t < \mu - |x|\}; \\ \Sigma_{\tau_1,\mu} &\stackrel{\text{def}}{=} \{(t,x) \in \mathbb{R}^{1+d} \mid t^2 - |x|^2 = \tau_1^2, t < \mu - |x|\}. \end{aligned}$$

Integrating the identity (2.2.8) with $M = 0$ for u^1 and $M = 1$ for u^i over ${}_{\mu}\mathcal{D}_{\tau_1}^{\tau_2}$, applying the divergence theorem, using the coercivity for $\int_{\Sigma_\tau} Q[\phi](\partial_t, \partial_\tau)\,\mathrm{dvol}_\tau$ afforded by Lemma 2.2.5, throwing out the boundary integrals along C_μ (which is possible because of the dominant energy condition [see Lemma 2.2.1]), and letting $\mu \to \infty$, one arrives to the desired inequality (3.5.3).

It is useful to work with the total energy of the coupled wave-Klein–Gordon system. We then schematically write

$$\mathcal{E}_\tau[\boldsymbol{u}] \stackrel{\text{def}}{=} \sqrt{\mathcal{E}_\tau^W[u^1]^2 + \sum_{i=1}^n \mathcal{E}_\tau^{KG}[u^i]^2}.$$

With this the fundamental estimate simplifies to

$$\mathcal{E}_{\tau_1}[\boldsymbol{u}]^2 - \mathcal{E}_{\tau_0}[\boldsymbol{u}]^2 \lesssim \int_{\tau_0}^{\tau_1} \int_{\Sigma_\tau} \Box_{\mathrm{m}} u^1 \partial_t u^1 + \langle \Box_{\mathrm{m}} \vec{u} - \vec{u}, \partial_t \vec{u} \rangle \ \mathrm{dvol}_{\Sigma_\tau} \,\mathrm{d}\tau. \tag{3.5.4}$$

In order to apply (3.5.3) to attain higher derivative estimates of $\boldsymbol{u}$, we commute the system (3.4.13) – (3.4.14) with the Lorentz boosts. It is useful to introduce a notation for higher order energies in order to close the bootstrap assumption in a systematic way. We define

$$\mathfrak{E}_k(\tau) \stackrel{\text{def}}{=} \tau^{-1/2} \|\boldsymbol{u}\|_{\mathcal{W}_{-1}^{k+1,2}} + \tau^{1/2} \|\partial_t \boldsymbol{u}\|_{\mathcal{W}_{-1}^{k,2}} + \tau^{-1/2} \|\vec{u}\|_{\mathcal{W}_1^{k,2}}. \tag{3.5.5}$$

We note that the second estimate (2.2.13) of Lemma 2.2.5 implies

$$\mathfrak{E}_k(\tau) \approx \sum_{|\alpha| \le k} \mathcal{E}_\tau[L^\alpha \boldsymbol{u}].$$

For convenience, we record the pointwise and integrated decay estimates of Chapter 2 in the notation adapted to $\boldsymbol{u}$:

Proposition 3.5.1. *For any $x \in \Sigma_\tau$, the following pointwise estimates hold:*

$$|\vec{u}(x)| \lesssim w_\tau(x)^{-3/2}\mathfrak{E}_2(\tau),$$
$$|L^i \boldsymbol{u}(x)| + \tau|\partial_t \boldsymbol{u}(x)| \lesssim w_\tau(x)^{-1/2}\mathfrak{E}_2(\tau).$$

The following Sobolev estimates hold for any $r \in [2,6]$:

$$\|\vec{u}\|_{\mathring{\mathcal{W}}_1^{k,r}(\Sigma_\tau)} \lesssim \tau^{1/r}\mathfrak{E}_k(\tau),$$
$$\|\boldsymbol{u}\|_{\mathring{\mathcal{W}}_{r/2-2}^{k+1,r}(\Sigma_\tau)} + \tau\|\partial_t \boldsymbol{u}\|_{\mathring{\mathcal{W}}_{r/2-2}^{k,r}(\Sigma_\tau)} \lesssim \tau^{1/r}\mathfrak{E}_k(\tau)^{\frac{6-r}{2r}} \cdot \mathfrak{E}_{k+1}(\tau)^{\frac{3r-6}{2r}}.$$

Proof. The estimates are an immediate consequence of Propositions 2.2.7, 2.2.11, 2.2.17 and the definition of $\mathfrak{E}_k(\tau)$. □

3.6 Global stability in the setting of TL

In this section we use the estimates recorded in the former in order to prove global existence to the following wave-Klein–Gordon system:

$$\begin{aligned} \Box_{\mathfrak{m}} u^1 &= -2\langle \vec{u}, \partial_t \vec{u}\rangle + (\vec{u})^3 + (\vec{u})^2\partial_t \boldsymbol{u} + \vec{u}\cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}), \\ \Box_{\mathfrak{m}} u^i - u^i &= 2u^i\partial_t u^1 + (\vec{u})^3 + (\vec{u})^2\partial_t \boldsymbol{u} + \vec{u}\cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}), \quad i = 2,\dots,n \end{aligned} \tag{3.6.1}$$

where here $(\vec{u})^3$, $(\vec{u})^2\partial_t \boldsymbol{u}$, and $\vec{u}\cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u})$ is an abuse of notation representing a linear combination of terms of the form

$$\begin{aligned} &u^m u^p u^q, \quad m,p,q \in \{2,\dots,n\}, \\ &u^m u^p \partial_t u^j, \qquad m,p \in \{2,\dots,n\}, \qquad j \in \{1,\dots,n\}, \\ &u^m \mathfrak{m}(\mathrm{d}u^j, \mathrm{d}u^k), \qquad m \in \{2,\dots,n\}, \quad j,k \in \{1,\dots,n\}. \end{aligned} \tag{3.6.2}$$

For our convenience, we will prescribe initial data at $t = 2$:

$$\boldsymbol{u}(2,x) = \boldsymbol{\phi}_0(x), \qquad \partial_t \boldsymbol{u}(2,x) = \boldsymbol{\varphi}_0(x).$$

Even though this system is a simplification of (3.4.13)–(3.4.14), it captures all of the analytical difficulties and extending the results to the full equations of motion is merely a matter of bookkeeping. Indeed, as the coefficients of (3.6.2) in the full system are of the form $\partial^{\leq 3}\Gamma(\ell, 0)$, they can be regarded as constant as a consequence of the manifold (M, g) having constant curvature[4]. Moreover, the higher ordered terms in (3.4.13)–(3.4.14) are

$$\left(|\vec{u}|^4 + |\vec{u}|^3|\partial_t \boldsymbol{u}| + |\vec{u}|^2|\mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u})|\right) f(\boldsymbol{u}, \partial\boldsymbol{u}),$$

where $f : \mathbb{R}^{n(d+2)} \to \mathbb{R}$ is an arbitrary smooth function. The standard argument, using the energy method, for either the stability problem or the local existence problem for nonlinear waves, handles the nonlinearities with the general prescription of "putting the highest order derivative factor in L^2 and the remainder in L^∞." As the L^∞ estimates we will be using are the pointwise bounds from Proposition 3.5.1, we see that higher order nonlinearities lead to *more* available decay, and hence add no difficulties when improving the bootstrap assumptions.

Our main theorem asserts that a geodesic wave map affinely parametrized by a time-like *linear* free wave is stable under small (in an appropriate Sobolev norm) perturbations, and that the perturbed solution stays within a small tubular neighborhood of the background geodesic.

Theorem 3.6.1. *For any $\gamma < 1/2$, there exists some ϵ_0 (which depends only on γ) such that whenever $\boldsymbol{\phi}_0, \boldsymbol{\varphi}_0$ are compactly supported in the ball of radius 1 centered at the origin satisfying*

$$\|\boldsymbol{\phi}_0\|_{H^3} + \|\boldsymbol{\varphi}_0\|_{H^2} < \epsilon_0,$$

[4] We now give a few more details explaining this fact for $\partial_m \Gamma^i_{jk}$; higher derivatives of the Christoffel symbols follow similarly. Similar analysis used to prove Lemma 3.4.4 and the geodesy of φ_{I} show that $\mathcal{L}_{\partial_1} \partial_m \Gamma^i_{jk} = 0$. Equivalently, $\partial_m \Gamma^i_{jk}$ is *constant along φ_I*. This can be interpreted geometrically as the flow map of ∂_1 is a *transvection* along the geodesic φ_{I}, see [O'N83, Definition 2.29]

there exists a unique solution $\boldsymbol{u} = (u^1, \vec{u})$ *to* (3.6.1) *that exists for all time* $t \geq 2$*. Furthermore, we have the following uniform estimates:*

$$|u^1| + \sum_{i=1}^{3} |L^i \boldsymbol{u}| \lesssim \tau^{\gamma} t^{-1/2}$$

$$|\vec{u}| \lesssim \tau^{\gamma} t^{-3/2}$$

$$|\partial_t \boldsymbol{u}| \lesssim \tau^{-1+\gamma} t^{-1/2}.$$

By standard local existence theory we can assume that for sufficiently small initial data, the solution $\boldsymbol{u}$ of (3.6.1) exists up to Σ_2. The breakdown criterion for wave and Klein–Gordon equations implies that so long as we can show that $|\vec{u}|$, $|L\boldsymbol{u}|$, $|\partial_t \boldsymbol{u}|$ remain bounded on Σ_τ for all $\tau > 2$, we can guarantee global existence of solutions. See [Sog08, Chapter 1, Theorem 4.3]. Proposition 3.5.1 implies that a sufficient condition for global existence are a priori estimates on the second order energies. The general approach is that of a bootstrap argument:

1. We will assume that, up to time $\tau_{\max} > 2$, that the energies $\mathfrak{E}_k(\tau)$ of the solution $\boldsymbol{u}$ and its derivatives $L^\alpha \boldsymbol{u}$ verify certain bounds.

2. Using Proposition 3.5.1, this gives L^∞ bounds on $\boldsymbol{u}$, and its derivatives of the form $L^\alpha \boldsymbol{u}$ and $\partial_t L^\alpha \boldsymbol{u}$.

3. We can then estimate the nonlinearity using these L^∞ estimates, which we then feed back into the energy inequality (3.5.4) to get an *updated* control on $\mathfrak{E}_k(\tau)$ for all $\tau \in [2, \tau_{\max}]$.

4. Finally, show for sufficiently small initial data sizes, that the updated control *improves* the original control, whereupon by the method of continuity the original bounds on $\mathfrak{E}_k(\tau)$ must hold for all $\tau \geq 2$, implying the desired global existence.

Since the Lorentz boosts L^i commute with the d'Alembertian $[L^i, \Box_{\mathfrak{m}}] = 0$, after applying (3.5.4) to $L^\alpha \boldsymbol{u} = (L^\alpha u^1, L^\alpha \vec{u})$ we see that we need to estimate the integrals

$$\int_{\Sigma_\tau} L^\alpha(\Box_{\mathfrak{m}} u^1)\partial_t L^\alpha u^1 + \langle L^\alpha(\Box_{\mathfrak{m}}\vec{u} - \vec{u}), \partial_t L^\alpha \vec{u}\rangle \ \mathrm{dvol}_{\Sigma_\tau} \tag{3.6.3}$$

for all tuples α with elements drawn from $\{1,2,3\}$ and length ≤ 2.

From the structure of (3.6.1), when $|\alpha| = 0$ we see a *complete* cancellation of the quadratic terms in (3.6.3):

$$-2\langle \vec{u}, \partial_t \vec{u}\rangle \partial_t u^1 + 2\langle \partial_t u^1 \vec{u}, \partial_t \vec{u}\rangle = 0. \tag{3.6.4}$$

Although this cancellation is unique to the case of $|\alpha| = 0$, for $|\alpha| = 1, 2$ we do see a cancellation of all of the *top order* derivative quadratic terms. We see for any tuple α with elements drawn from $\{1,2,3\}$

$$\begin{aligned}\Big|L^\alpha\Big(-2\langle \vec{u}, \partial_t \vec{u}\rangle\Big)\partial_t L^\alpha u^1 + \Big\langle L^\alpha(2\partial_t u^1 \vec{u}), \partial_t L^\alpha \vec{u}\Big\rangle\Big| &\lesssim \\ &\Big|-\langle \vec{u}, L^\alpha \partial_t \vec{u}\rangle \partial_t L^\alpha u^1 + L^\alpha \partial_t u^1 \langle \vec{u}, \partial_t L^\alpha \vec{u}\rangle\Big| \\ &+ \sum_{\substack{|\beta|+|\gamma|\le|\alpha| \\ |\beta|\neq|\alpha|}} \Big|\langle L^\gamma \vec{u}, L^\beta \partial_t \vec{u}\rangle \partial_t L^\alpha u^1 + L^\beta \partial_t u^1 \langle L^\gamma \vec{u}, \partial_t L^\alpha \vec{u}\rangle\Big|.\end{aligned} \tag{3.6.5}$$

Using the commutator algebra properties

$$[L^i, \partial_t] = -\partial_{x^i} = -\frac{1}{t}L^i + \frac{x^i}{t}\partial_t, \tag{3.6.6}$$

we see a cancellation of the top order terms in the first term on the right hand side of (3.6.5). Consequently, the quadratic terms of (3.6.3) can be estimated schematically as

$$\lesssim \int_{\Sigma_\tau} \sum_{\substack{|\beta|+|\gamma|\le|\alpha| \\ |\beta|\neq\alpha}} |L^\gamma \vec{u} \partial_t L^\beta \boldsymbol{u} \partial_t L^\alpha \boldsymbol{u}| + \sum_{|\beta|+|\gamma|\le|\alpha|} w_\tau^{-1} |L^\gamma \vec{u} L^\beta \boldsymbol{u} \partial_t L^\alpha \boldsymbol{u}| \ \mathrm{dvol}_{\Sigma_\tau} \tag{3.6.7}$$

where we repeatedly used (3.6.6) and (2.1.8). We can now estimate the quadratic terms of (3.6.3).

Proposition 3.6.2 (Quadratic energy estimates)**.** *Let $\alpha \neq 0$ be an m-tuple*[5] *with elements drawn from* $\{1,2,3\}$*. Then*

$$\int_{\Sigma_\tau} \Big| L^\alpha\Big(-2\langle\vec{u},\partial_t\vec{u}\rangle\Big)\partial_t L^\alpha u^1 + \Big\langle L^\alpha(2\partial_t u^1\vec{u}),\partial_t L^\alpha\vec{u}\Big\rangle\Big| \,\mathrm{dvol}_{\Sigma_\tau} \lesssim$$

$$\begin{cases} \tau^{-3/2}\mathcal{E}_1^2\cdot\mathcal{E}_2 & m=1, \\ \tau^{-3/2}\mathcal{E}_2^3+\tau^{-1}\mathcal{E}_1\cdot\mathcal{E}_2^2+\tau^{-1}\mathcal{E}_1^2\cdot\mathcal{E}_2 & m=2. \end{cases}$$

Proof. Throughout this proof we use the simple inequality $w_\tau^{-1}\leq\tau^{-1}$. We prove the estimate for the case $m=1$ first. In this case the top ordered derivative terms of (3.6.7) that we need to estimate are of the form

$$\int_{\Sigma_\tau} |L\vec{u}\partial_t\boldsymbol{u}\partial_t L\boldsymbol{u}| + w_\tau^{-1}|\vec{u}L\boldsymbol{u}\partial_t L\boldsymbol{u}| \ \mathrm{dvol}_{\Sigma_\tau}$$

(the estimates for the lower ordered terms will of course be controlled by the top ones). Here it is understood that L can be any of the boosts L^i. For the first term, we estimate the $\partial_t\boldsymbol{u}$ factor by the pointwise estimates in Proposition 3.5.1 and Hölder's inequality on the rest of them

$$\begin{aligned}\int_{\Sigma_\tau} |L\vec{u}\partial_t\boldsymbol{u}\partial_t L\boldsymbol{u}| \ \mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-3/2}\mathcal{E}_2(\tau)\int_{\Sigma_\tau}|L\vec{u}\ w_\tau^{1/2}|\cdot|\partial_t L\boldsymbol{u}\ w_\tau^{-1/2}| \ \mathrm{dvol}_{\Sigma_\tau} \\ &\leq \tau^{-3/2}\mathcal{E}_2(\tau)\cdot(\mathcal{E}_1(\tau))^2.\end{aligned}$$

For the second term, we control the $\vec{u}$ factor by the pointwise estimates and use Hölder's inequality on the rest

$$\begin{aligned}\int_{\Sigma_\tau} w_\tau^{-1}|\vec{u}L\boldsymbol{u}\partial_t L\boldsymbol{u}| \ \mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-3/2}\mathcal{E}_2(\tau)\int_{\Sigma_\tau}|L\boldsymbol{u}\ w_\tau^{-1/2}|\cdot|\partial_t L\boldsymbol{u}\ w_\tau^{-1/2}| \ \mathrm{dvol} \\ &\leq \tau^{-3/2}\mathcal{E}_2(\tau)\cdot(\mathcal{E}_1(\tau))^2.\end{aligned}$$

This concludes the proof for $m=1$.

[5]The case $m=0$ does not need to be controlled due to (3.6.4).

For $m = 2$, the terms from (3.6.7) are

$$\int_{\Sigma_\tau} |LL\vec{u}\partial_t \boldsymbol{u}\partial_t LL\boldsymbol{u}| + w_\tau^{-1}|\vec{u}LL\boldsymbol{u}\partial_t LL\boldsymbol{u}|$$
$$+ |L\vec{u}\partial_t L\boldsymbol{u}\partial_t LL\boldsymbol{u}| + w_\tau^{-1}|L\vec{u}L\boldsymbol{u}\partial_t LL\boldsymbol{u}| \ \mathrm{dvol}_{\Sigma_\tau}.$$

Again, the estimates for all lower ordered terms can be controlled by the estimates of these. Here it is understood that LL is any arbitrary second order tangential derivative $L^i L^j$. The first two terms are bounded by

$$\tau^{-3/2}(\mathfrak{E}_2(\tau))^3$$

using the same techniques as $m = 1$ (estimating the lowest ordered terms in L^∞ and the rest by the energies after using Hölder's). The other two terms cannot be treated with the same techniques. Even though $|L\vec{u}| + |L\boldsymbol{u}|$ can be bounded by $w_\tau^{-1/2}\mathfrak{E}_2(\tau)$, this decay is too weak to improve the bootstrap assumptions that we will make. On the other hand, we can get stronger decay for the third term above by estimating $|L\vec{u}| \le w_\tau^{-3/2}\mathfrak{E}_3(\tau)$. This is not helpful to us because $\mathfrak{E}_3(\tau)$ requires square integrability of *four* derivatives of $\boldsymbol{u}$ (recall that we want to solve the Cauchy problem for (3.6.1) using data in H^3).

We instead appeal to the interpolated Sobolev estimates in Proposition 3.5.1 with $r = 3, 6$ to control the third and fourth terms above. We see

$$\begin{aligned}\int_{\Sigma_\tau} |L\vec{u}\partial_t L\boldsymbol{u}\partial_t LL\boldsymbol{u}| \ \mathrm{dvol}_{\Sigma_\tau} &= \int_{\Sigma_\tau} |L\vec{u}\, w_\tau^{1/3}| \cdot |\partial_t L\boldsymbol{u}\, w_\tau^{1/6}| \cdot |\partial_t LL\boldsymbol{u}\, w_\tau^{-1/2}| \ \mathrm{dvol}_{\Sigma_\tau} \\ &\le \|\vec{u}\|_{\mathring{\mathcal{W}}_1^{1,3}} \|\partial_t \boldsymbol{u}\|_{\mathring{\mathcal{W}}_1^{1,6}} \|\partial_t \boldsymbol{u}\|_{\mathring{\mathcal{W}}_{-1}^{2,2}} \\ &\le \tau^{-1}\mathfrak{E}_1(\tau) \cdot (\mathfrak{E}_2(\tau))^2. \end{aligned} \tag{3.6.8}$$

Similarly, we see

$$\begin{aligned}\int_{\Sigma_\tau} w_\tau^{-1}|L\vec{u}L\boldsymbol{u}\partial_t LL\boldsymbol{u}|\ \mathrm{dvol}_{\Sigma_\tau} &\leq \tau^{-1}\int_{\Sigma_\tau}|L\vec{u}\, w_\tau^{1/3}|\cdot|L\boldsymbol{u}\, w_\tau^{1/6}|\cdot|\partial_t LL\boldsymbol{u}\, w_\tau^{-1/2}|\ \mathrm{dvol}_{\Sigma_\tau}\\ &\leq \tau^{-1}\|\vec{u}\|_{\mathring{\mathcal{W}}_1^{1,3}}\|\boldsymbol{u}\|_{\mathring{\mathcal{W}}_1^{1,6}}\|\partial_t\boldsymbol{u}\|_{\mathring{\mathcal{W}}_{-1}^{2,2}}\\ &\leq \tau^{-1}(\mathfrak{E}_1(\tau))^2\cdot(\mathfrak{E}_2(\tau)).\end{aligned} \tag{3.6.9}$$

This concludes the proof of the proposition. □

Remark 3.6.3. The expression on the right hand side of (3.6.9) would allow us to close our energy estimates with only a log loss, see Proposition 3.6.6. The borderline terms that we need to deal with are in fact in (3.6.8).

Estimating the cubic terms in (3.6.1) we identify the integrals that we have to estimate are

$$\int_{\Sigma_\tau}\Big(L^\beta\vec{u}\,L^\gamma\vec{u}\,L^\sigma\vec{u}+L^\beta\vec{u}\cdot\mathfrak{m}(\mathrm{d}L^\gamma\boldsymbol{u},\mathrm{d}L^\sigma\boldsymbol{u})+L^\beta\vec{u}\,L^\gamma\vec{u}\,L^\sigma\partial_t\boldsymbol{u}\Big)\cdot\partial_t L^\alpha\boldsymbol{u}\,\mathrm{dvol}_{\Sigma_\tau}$$

for $|\beta|+|\gamma|+|\sigma|=|\alpha|$. Here we implicitly used that vector fields act on scalars by Lie differentiation, that $\mathfrak{m}$ is invariant under the Lorentz boosts L^i, and that exterior differentiation commutes with Lie differentiation.

Proposition 3.6.4 (Cubic energy estimates)**.** *Let α be an m-tuple with elements drawn from $\{1,2,3\}$. Then*

$$\int_{\Sigma_\tau}\Big|L^\alpha((\vec{u}))^3+L^\alpha((\vec{u})^2\partial_t\boldsymbol{u})+L^\alpha(\vec{u}\cdot\mathfrak{m}(\mathrm{d}\boldsymbol{u},\mathrm{d}\boldsymbol{u}))\Big|\cdot|\partial_t L^\alpha\boldsymbol{u}|\ \mathrm{dvol}_{\Sigma_\tau}\lesssim$$

$$\begin{cases}\tau^{-3}\mathfrak{E}_m^2\cdot\mathfrak{E}_2^2 & m=0,1\\ \tau^{-3}\mathfrak{E}_2^4+\tau^{-2}\mathfrak{E}_0^{1/4}\cdot\mathfrak{E}_1\cdot\mathfrak{E}_2^{11/4}+\tau^{-3/2}\mathfrak{E}_1^2\cdot\mathfrak{E}_2^2+\tau^{-5/2}\mathfrak{E}_1\cdot\mathfrak{E}_2^3 & m=2\end{cases}$$

Proof. Let us treat the terms with $(\vec{u})^3$ first. When $m = 0$, we control two of the factors by the pointwise estimates:

$$\int_{\Sigma_\tau} |(\vec{u})^3 \cdot \partial_t \boldsymbol{u}| \, \mathrm{dvol}_{\Sigma_\tau} \lesssim \tau^{-3} \mathcal{E}_2(\tau)^2 \int_{\Sigma_\tau} |\vec{u}\, w_\tau^{1/2}| \cdot |\partial_t \boldsymbol{u}\, w_\tau^{-1/2}| \, \mathrm{dvol}_{\Sigma_\tau}$$
$$\lesssim \tau^{-3} \mathcal{E}_0(\tau)^2 \cdot \mathcal{E}_2(\tau)^2,$$

as desired. For $m = 1$, the same proof follows by controlling the two factors that are *not* differentiated by the pointwise estimates (note that the density is $|L\vec{u}(\vec{u})^2 \partial_t L\boldsymbol{u}|$). When $m = 2$, this can again be used to bound the terms of the form

$$\int_{\Sigma_\tau} |LL\vec{u} \cdot (\vec{u})^2 \cdot \partial_t LL\boldsymbol{u}| \, \mathrm{dvol}_{\Sigma_\tau} \lesssim \tau^{-3} \mathcal{E}_2(\tau)^4.$$

For the other cases, we couple the pointwise estimates *and* the interpolated GNS estimates of Proposition 3.5.1 to find

$$\int_{\Sigma_\tau} |(L\vec{u})^2 \cdot \vec{u} \cdot \partial_t LL\boldsymbol{u}| \, \mathrm{dvol}_{\Sigma_\tau} \lesssim \tau^{-3/2} \mathcal{E}_2 \int_{\Sigma_\tau} |L\vec{u}\, w_\tau^{1/3}| \cdot |L\vec{u}\, w_\tau^{1/6}| \cdot |\partial_t LL\boldsymbol{u}\, w_\tau^{-1/2}| \, \mathrm{dvol}_{\Sigma_\tau}$$
$$\lesssim \tau^{-3/2} \mathcal{E}_1(\tau)^2 \cdot \mathcal{E}_2(\tau)^2.$$

Next we control the $(\vec{u})^2 \partial_t \boldsymbol{u}$ terms. For $m = 0$, we control one $\partial_t \boldsymbol{u}$ and one Klein–Gordon term by the energy:

$$\int_{\Sigma_\tau} |(\vec{u})^2 \partial_t \boldsymbol{u} \cdot \partial_t \boldsymbol{u}| \, \mathrm{dvol}_{\Sigma_\tau} \lesssim \tau^{-3} \mathcal{E}_2 \int_{\Sigma_\tau} |\vec{u}\, w_\tau^{1/2}| \cdot |\partial_t \boldsymbol{u}\, w_\tau^{-1/2}| \, \mathrm{dvol}_{\Sigma_\tau}$$
$$\lesssim \tau^{-3} \mathcal{E}_0(\tau)^2 \cdot \mathcal{E}_2(\tau)^2.$$

For $m = 1$, the same technique is used to bound

$$\int_{\Sigma_\tau} |\vec{u}\, L\vec{u}\, \partial_t u \cdot \partial_t L u| \mathrm{dvol}_{\Sigma_\tau} \lesssim \tau^{-3} \mathcal{E}_1(\tau)^2 \mathcal{E}_2(\tau)^2.$$

When the derivative hits the $\partial_t \boldsymbol{u}$ factor we sacrifice some of the decay given by the Klein–Gordon terms[6]:

$$\int_{\Sigma_\tau} |(\vec{u})^2 \partial_t L\boldsymbol{u} \cdot \partial_t L\boldsymbol{u}| \,\mathrm{dvol}_{\Sigma_\tau} \lesssim \tau^{-2} \mathcal{E}_2^2 \int_{\Sigma_\tau} |\partial_t L\boldsymbol{u}\, w_\tau^{-1/2}| \cdot |\partial_t L\boldsymbol{u}\, w_\tau^{-1/2}| \,\mathrm{dvol}_{\Sigma_\tau}$$
$$\lesssim \tau^{-2} \mathcal{E}_2(\tau)^2 \cdot \|\partial_t u\|^2_{\mathring{\mathcal{W}}^{1,2}_{-1}}$$
$$\lesssim \tau^{-3} \mathcal{E}_1(\tau)^2 \cdot \mathcal{E}_2(\tau)^2.$$

For $m = 2$ the densities we need to estimate are $\partial_t LL\boldsymbol{u}$ multiplied by [7]

$$\vec{u}\, LL\vec{u}\, \partial_t \boldsymbol{u}, \qquad (L\vec{u})^2 \partial_t \boldsymbol{u}, \qquad \vec{u}\, L\vec{u}\, \partial_t L\vec{u}, \qquad (\vec{u})^2 \partial_t LL\boldsymbol{u}. \tag{3.6.10}$$

For the first density we estimate the undifferentiated terms by the energies as we did for $m = 0$ and $m = 1$ to see

$$\int_{\Sigma_\tau} |\vec{u}\, LL\vec{u}\, \partial_t \boldsymbol{u} \cdot \partial_t LL\boldsymbol{u}| \mathrm{dvol}_{\Sigma_\tau} \lesssim \tau^{-3} \mathcal{E}_2(\tau)^4.$$

The second density of (3.6.10) is treated with the interpolation inequalities after using the pointwise estimate to control $\partial_t \boldsymbol{u}$:

$$\int_{\Sigma_\tau} |(L\vec{u})^2 \partial_t \boldsymbol{u} \cdot \partial_t LL\boldsymbol{u}| \,\mathrm{dvol}_{\Sigma_\tau} \lesssim \tau^{-3/2} \mathcal{E}_2 \int_{\Sigma_\tau} |L\vec{u}\, w_\tau^{1/3}| \cdot |L\vec{u}\, w_\tau^{1/6}| \cdot |\partial_t LL\boldsymbol{u}\, w_\tau^{-1/2}| \,\mathrm{dvol}_{\Sigma_\tau}$$
$$\lesssim \tau^{-3/2} \mathcal{E}_2 \|\vec{u}\|_{\mathring{\mathcal{W}}^{1,3}_1} \|\vec{u}\|_{\mathring{\mathcal{W}}^{1,6}_1} \|\partial_t \boldsymbol{u}\|_{\mathring{\mathcal{W}}^{2,2}_{-1}}$$
$$\lesssim \tau^{-3/2} \mathcal{E}_1(\tau)^2 \mathcal{E}_2(\tau)^2.$$

[6] Of course, there are lower-ordered terms which appear as a consequence of commuting the derivative: $L^i \partial_t \boldsymbol{u} = \partial_t L^i \boldsymbol{u} - w_\tau^{-1} L^i \boldsymbol{u} + \frac{x^i}{w_\tau} \partial_t \boldsymbol{u}$. One can check that the energies of these commuted terms are bounded by $\tau^{-3} \mathcal{E}_0 \mathcal{E}_1 \mathcal{E}_2^2$. We drop these lower ordered energies because will of course be controlled by $\tau^{-3} \mathcal{E}_1^2 \mathcal{E}_2^2$.

[7] Again, there are lower ordered terms that rise from commuting L^i and ∂_t. We drop these energies because one can check that they will all be controlled by the energies of $(\vec{u})^2\, \partial_t LL\boldsymbol{u}$.

The third density is treated similarly:

$$\begin{aligned}\int\limits_{\Sigma_\tau} |\vec{u}\, L\vec{u}\, \partial_t L\vec{u} \cdot \partial_t LL\boldsymbol{u}|\ \mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-3/2}\mathfrak{E}_2 \int\limits_{\Sigma_\tau} |L\vec{u}\, w_\tau^{1/3}| \cdot |\partial_t L\boldsymbol{u}\, w_\tau^{1/6}| \cdot |\partial_t LL\boldsymbol{u}\, w_\tau^{-1/2}|\ \mathrm{dvol}_{\Sigma_\tau} \\ &\lesssim \tau^{-3/2}\mathfrak{E}_2(\tau) \cdot \|\vec{u}\|_{\mathring{\mathcal{W}}_1^{1,3}} \cdot \|\partial_t \boldsymbol{u}\|_{\mathring{\mathcal{W}}_1^{1,6}} \cdot \|\partial_t \boldsymbol{u}\|_{\mathring{\mathcal{W}}_{-1}^{2,2}} \\ &\lesssim \tau^{-5/2}\mathfrak{E}_1(\tau)\mathfrak{E}_2(\tau)^3.\end{aligned}$$

The last case of (3.6.10) is treated by controlling the two Klein–Gordon factors by the pointwise estimates

$$\begin{aligned}\int\limits_{\Sigma_\tau} |(\vec{u})^2 \partial_t LL\boldsymbol{u} \cdot \partial_t LL\boldsymbol{u}| \mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-2}\mathfrak{E}_2^2 \int\limits_{\Sigma_\tau} |\partial_t LL\boldsymbol{u}\, w_\tau^{-1/2}| \cdot |\partial_t LL\boldsymbol{u}\, w_\tau^{-1/2}|\ \mathrm{dvol}_{\Sigma_\tau} \\ &\lesssim \tau^{-2}\mathfrak{E}_2(\tau)^2 \cdot \|\partial_t \boldsymbol{u}\|^2_{\mathring{\mathcal{W}}_{-1}^{2,2}} \\ &\lesssim \tau^{-3}\mathfrak{E}_2(\tau)^4.\end{aligned}$$

We finally treat the $\vec{u} \cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u})$ terms. Note firstly that the second equality in (3.6.6) implies the estimate

$$|\mathfrak{m}(\mathrm{d}\psi_1, \mathrm{d}\psi_2)| \le \frac{1}{\tau^2}|L\psi_1| \cdot |L\psi_2| + |\partial_t \psi_1| \cdot |\partial_t \psi_2|$$

for any scalars $\psi_1,\ \psi_2$. For $m = 0$ the pointwise estimates imply

$$\begin{aligned}\int\limits_{\Sigma_\tau} |\vec{u} \cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}) \cdot \partial_t \boldsymbol{u}| \mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-3}\mathfrak{E}_2(\tau)^2 \int\limits_{\Sigma_\tau} |\vec{u}\, w_\tau^{1/2}| \cdot |\partial_t \boldsymbol{u}\, w_\tau^{-1/2}|\ \mathrm{dvol}_{\Sigma_\tau} \\ &\lesssim \tau^{-3}\mathfrak{E}_0(\tau)^2\mathfrak{E}_2(\tau)^2.\end{aligned}$$

For $m = 1$ we similarly see

$$\int\limits_{\Sigma_\tau} |L\vec{u} \cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}) \cdot \partial_t L\boldsymbol{u}| \mathrm{dvol}_{\Sigma_\tau} \lesssim \tau^{-3}\mathfrak{E}_1(\tau)^2\mathfrak{E}_2(\tau)^2.$$

When the derivative hits the null form factor the density is $\vec{u}\cdot\mathfrak{m}(\mathrm{d}Lu,\mathrm{d}u)\cdot\partial_t L\boldsymbol{u}$. We can then use the improved Klein–Gordon decay $|\vec{u}|\lesssim w_\tau^{-3/2}\mathfrak{E}_2$ to estimate

$$\begin{aligned}\int\limits_{\Sigma_\tau}|\vec{u}\cdot\mathfrak{m}(\mathrm{d}L\boldsymbol{u},\mathrm{d}\boldsymbol{u})\cdot\partial_t L\boldsymbol{u}|\mathrm{dvol} &\lesssim \tau^{-1}\mathfrak{E}_2\int\limits_{\Sigma_\tau}\left(\tau^{-2}|LL\boldsymbol{u}|\cdot|L\boldsymbol{u}|+|\partial_t L\boldsymbol{u}|\cdot|\partial_t\boldsymbol{u}|\right)\frac{|\partial_t L\boldsymbol{u}|}{w_\tau^{1/2}}\ \mathrm{dvol}\\ &\lesssim \tau^{-2}\mathfrak{E}_2^2\int\limits_{\Sigma_\tau}\left(\frac{|LL\boldsymbol{u}|}{\tau w_\tau^{1/2}}+\frac{|\partial_t L\boldsymbol{u}|}{w_\tau^{1/2}}\right)\frac{|\partial_t L\boldsymbol{u}|}{w_\tau^{1/2}}\ \mathrm{dvol}\\ &\lesssim \tau^{-3}\mathfrak{E}_1^2\mathfrak{E}_2^2.\end{aligned}$$

Replicating the previous estimates, when $m=2$,

$$\int\limits_{\Sigma_\tau}\left|LL\vec{u}\cdot\mathfrak{m}(\mathrm{d}\boldsymbol{u},\mathrm{d}\boldsymbol{u})+\vec{u}\cdot\mathfrak{m}(\mathrm{d}LL\boldsymbol{u},\mathrm{d}\boldsymbol{u})\right|\cdot|\partial_t LL\boldsymbol{u}|\mathrm{dvol}\lesssim\tau^{-3}\mathfrak{E}_2^4.$$

The remaining term

$$\int\limits_{\Sigma_\tau}|L\vec{u}\cdot\mathfrak{m}(\mathrm{d}L\boldsymbol{u},\mathrm{d}\boldsymbol{u})\partial_t LL\boldsymbol{u}|\ \mathrm{dvol}_{\Sigma_\tau}$$

can't be treated in the same way because the improved decay from the Klein–Gordon term comes at a loss of one derivative: $|L\vec{u}|\lesssim w_\tau^{-3/2}\mathfrak{E}_3$. We must then rely on the weaker estimate $|L\vec{u}|\lesssim w_\tau^{-1/2}\mathfrak{E}_2$ and remedy this loss with the interpolated GNS estimates from Proposition 3.5.1 with $r=4$:

$$\begin{aligned}\int\limits_{\Sigma_\tau}|L\vec{u}\cdot\mathfrak{m}(\mathrm{d}L\boldsymbol{u},\mathrm{d}\boldsymbol{u})\cdot\partial_t L\boldsymbol{u}|\mathrm{dvol} &\lesssim \mathfrak{E}_2\int\limits_{\Sigma_\tau}\left(\tau^{-2}|LL\boldsymbol{u}|\cdot|L\boldsymbol{u}|+|\partial_t L\boldsymbol{u}|\cdot|\partial_t\boldsymbol{u}|\right)\frac{\partial_t LL\boldsymbol{u}}{w_\tau^{1/2}}\ \mathrm{dvol}\\ &\lesssim \tau^{-2}\mathfrak{E}_0^{1/4}\cdot\mathfrak{E}_1^{3/4}\cdot\mathfrak{E}_1^{1/4}\cdot\mathfrak{E}_2^{3/4}\cdot\mathfrak{E}_2.\end{aligned}$$

and the proposition follows. □

Using (3.5.5), we have as an immediate corollary of Propositions 3.6.2 and 3.6.4 the following *a priori* estimates:

Corollary 3.6.5.

$$\mathfrak{E}_0(\tau_1)^2 - \mathfrak{E}_0(\tau_0)^2 \lesssim \int_{\tau_0}^{\tau_1} \tau^{-3}\mathfrak{E}_0^2\mathfrak{E}_2^2 \,\mathrm{d}\tau \tag{3.6.11}$$

$$\mathfrak{E}_1(\tau_1)^2 - \mathfrak{E}_1(\tau_0)^2 \lesssim \int_{\tau_0}^{\tau_1} \tau^{-3/2}\mathfrak{E}_1^2\mathfrak{E}_2 + \tau^{-3}\mathfrak{E}_1^2\mathfrak{E}_2^2 \,\mathrm{d}\tau \tag{3.6.12}$$

$$\mathfrak{E}_2(\tau_1)^2 - \mathfrak{E}_2(\tau_0)^2 \lesssim \int_{\tau_0}^{\tau_1} \tau^{-1}\mathfrak{E}_1\mathfrak{E}_2\Big(\mathfrak{E}_1 + \mathfrak{E}_2\Big) + \tau^{-3/2}\mathfrak{E}_2^2\Big(\mathfrak{E}_2 + \mathfrak{E}_1\mathfrak{E}_2 + \mathfrak{E}_1^2\Big) \tag{3.6.13}$$

$$+ \tau^{-3}\mathfrak{E}_2^4 + \tau^{-2}\mathfrak{E}_0^{1/4}\cdot\mathfrak{E}_1\cdot\mathfrak{E}_2^{11/4}\,\mathrm{d}\tau. \tag{3.6.14}$$

These estimates imply the following bootstrap estimate.

Proposition 3.6.6. *Assume that the initial data satisfy*

$$\mathfrak{E}_2(2) \le \epsilon \tag{3.6.15}$$

and that for some $\tau_{max} > 2$ the bootstrap assumptions

$$\begin{cases} \mathfrak{E}_0(\tau) \le \delta \\ \mathfrak{E}_1(\tau) \le \delta \\ \mathfrak{E}_2(\tau) \le \delta\tau^{\gamma} \end{cases} \tag{3.6.16}$$

hold for all $\tau \in [2, \tau_{max}]$ and some $\delta < 1$, $\gamma \ll 1$. Then there exists a constant C depending only on γ such that the improved estimates

$$\begin{cases} \mathfrak{E}_0(\tau) \le \epsilon + C\delta^{3/2} \\ \mathfrak{E}_1(\tau) \le \epsilon + C\delta^{3/2} \\ \mathfrak{E}_2(\tau) \le \epsilon + C\delta^{3/2}\tau^{\gamma} \end{cases} \tag{3.6.17}$$

hold for all $\tau \in [2, \tau_{max}]$.

Proof. Improving the estimate for $\mathfrak{E}_0$ follows from (3.6.11) after noting that

$$\int_2^\tau \sigma^{-3}\mathfrak{E}_0(\sigma)^2 \cdot \mathfrak{E}_2(\sigma)^2 \, \mathrm{d}\sigma \le \delta^4 \int_2^\tau \sigma^{-3+2\gamma} \, \mathrm{d}\sigma \le \delta^4 \int_2^\infty \sigma^{-3+2\gamma} \, \mathrm{d}\sigma \le C\delta^3.$$

Similarly, the estimate for $\mathfrak{E}_1$ follows from (3.6.12) because $\sigma^{-3/2+\gamma}$ is integrable for $\sigma \in [2,\infty)$ provided that $\gamma < 1/2$.

We begin to improve the bootstrap $\mathfrak{E}_2$ by controlling the first two terms in the right hand side of (3.6.13), which are bounded by

$$\delta^3 \int_2^\tau \sigma^{-1+2\gamma} \, \mathrm{d}\sigma \le C\delta^3\tau^{2\gamma}.$$

The rest of the terms are all bounded by

$$\delta^3 \int_2^\infty \sigma^{-3/2+3\gamma} \, \mathrm{d}\sigma \le C\delta^3\tau^{2\gamma},$$

provided that $\gamma < 1/2$. We now consider γ fixed once and for all. □

As a consequence of the improved estimates, if we choose $\delta \le (4C)^{-1/2}$ and then $\epsilon < \delta/4$, we we conclude

$$\begin{cases} \mathfrak{E}_0(\tau) \le \frac{1}{2}\delta \\ \mathfrak{E}_1(\tau) \le \frac{1}{2}\delta \\ \mathfrak{E}_2(\tau) \le \frac{1}{2}\delta\tau^\gamma \end{cases}$$

In this case the global existence part of Theorem 3.6.1 follows by a continuity argument, and the decay estimates follow from an application of the pointwise estimates of Proposition 3.5.1 and these energy bounds.

3.7 Global stability in the setting of SL

In this last section we use the tools from Section **??** to prove stability of the totally geodesic background $\varphi_{\mathrm{I}} \circ \varphi_{\mathrm{S}}$ in the case that the target has positive curvature, i.e. (3.4.15) and (3.4.16). With the notations introduced in the previous section, we reduce our attention to

$$\begin{aligned}\Box_{\mathfrak{m}} u^1 &= 2\langle \vec{u}, \partial_{x^1} \vec{u} \rangle + (\vec{u})^3 + \vec{u} \cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}) + (\vec{u})^2 \cdot \partial_{x^1} \boldsymbol{u}, \\ \Box_{\mathfrak{m}} u^i - u^i &= -2u^i \partial_{x^1} u^1 + (\vec{u})^3 + \vec{u} \cdot \mathfrak{m}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}) + (\vec{u})^2 \cdot \partial_{x^1} \boldsymbol{u}, \quad i = 2, \dots, n\end{aligned} \tag{3.7.1}$$

With ∂_{x^1} replaced by ∂_t on the right hand side, the system above is the same with the negative curvature case (3.6.1). Employing

$$\partial_{x^i} = \frac{1}{t} L^i - \frac{x^i}{t} \partial_t, \tag{3.7.2}$$

what we can prove is:

Theorem 3.7.1. *Under the same assumptions, the results of Theorem 3.6.1 also apply to the system* (3.7.1).

Proof. It suffices to obtain similar estimates as those in Propositions 3.6.2 and 3.6.4, then the theorem 3.7.1 follows similarly from Corollary 3.6.5 and Proposition 3.6.6. We first deal with the quadratic terms. We decompose from (3.7.2) the quadratic terms into two parts, i.e. $Q(m) = Q_1(m) + Q_2(m)$ with

$$\begin{aligned}Q_1(m) &\overset{\text{def}}{=} \int_{\Sigma_\tau} \left[2L^\alpha \left\langle \vec{u}, t^{-1} L^1 \vec{u} \right\rangle \partial_t L^\alpha u^1 - 2 \left\langle L^\alpha (\vec{u}\, t^{-1} L^1 u^1), \partial_t L^\alpha \vec{u} \right\rangle \right] \mathrm{dvol}_{\Sigma_\tau}; \\ Q_2(m) &\overset{\text{def}}{=} \int_{\Sigma_\tau} \left[-2L^\alpha \left\langle \vec{u}, t^{-1} x^1 \partial_t \vec{u} \right\rangle \partial_t L^\alpha u^1 + 2 \left\langle L^\alpha (\vec{u}\, t^{-1} x^1 \partial_t u^1), \partial_t L^\alpha \vec{u} \right\rangle \right] \mathrm{dvol}_{\Sigma_\tau}\end{aligned}$$

for an m-tuple α with entries in $\{1, 2, 3\}$. The Q_2 term has the same structure with the quadratic nonlinearities for the negative curvature case presented in previous section, with the introduction of the factor $t^{-1} x^1 = w_\tau^{-1} x^1$. In particular, the top order terms can be canceled. As we will see, the boosts L^i acting on $t^{-1} x^1$ only contribute lower order terms because of (2.1.8).

On the other hand, the top order of $Q_1(m)$ can not be cancelled but we can utilize the extra decay of t^{-1}. We claim the quadratic terms can be bounded as

$$|Q(m)| \lesssim \begin{cases} \tau^{-3/2}\Big(\mathcal{E}_2(\tau)\mathcal{E}_m(\tau)^2\Big), & \text{if } m = 0, 1; \\ \tau^{-1}\mathcal{E}_1(\tau)\mathcal{E}_2(\tau)^2 + \tau^{-3/2}\mathcal{E}_2(\tau)^3, & \text{if } m = 2. \end{cases} \tag{3.7.3}$$

Besides the inequality $w_\tau^{-1} \lesssim \tau^{-1}$, we shall also use the identity $t = w_\tau$ on the surface Σ_τ. As Q_2 can be dealt with in the same way as Proposition (3.6.2), we only provide the proof for $Q_1(m)$. The case $m = 0$ is straightforward as we control $\vec{u}$ using the pointwise estimates of Proposition 3.5.1 and the rest of the vectors using Hölder's inequality. For the case $m = 1$ we need to estimate

$$\int_{\Sigma_\tau} \Big(w_\tau^{-1}|L\vec{u}|^2 + w_\tau^{-1}|\vec{u}||LL\boldsymbol{u}| + |L^\alpha(t^{-1})||\vec{u}||L\boldsymbol{u}|\Big)|\partial_t L\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau}. \tag{3.7.4}$$

Choosing the weights appropriately and applying Hölder's inequality imply

$$\begin{aligned}\int_{\Sigma_\tau} w_\tau^{-1}|L\vec{u}|^2|\partial_t L\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-3/2}\mathcal{E}_2(\tau)\int_{\Sigma_\tau} w_\tau^{1/2}|L\vec{u}|w_\tau^{-1/2}|\partial_t L\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} \\ &\lesssim \tau^{-3/2}\mathcal{E}_2(\tau)\mathcal{E}_1(\tau)^2.\end{aligned}$$

Here we also bounded L^∞-norm of $L\vec{u}$ through Proposition 3.5.1. With a use of the definition of L^i we have

$$L^i(t^{-1}) = -\frac{x^i}{t^2}$$

which on the surface Σ_τ admits an upper bound w_τ^{-1}, and hence implies

$$\begin{aligned}\int_{\Sigma_\tau} |L^\alpha(t^{-1})||\vec{u}||L\boldsymbol{u}||\partial_t L\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-3/2}\mathcal{E}_2(\tau)\int_{\Sigma_\tau} w_\tau^{-1/2}|L\boldsymbol{u}|w_\tau^{-1/2}|\partial_t L\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} \\ &\lesssim \tau^{-3/2}\mathcal{E}_2(\tau)\mathcal{E}_1(\tau)^2.\end{aligned}$$

Here we also applied the L^∞-bound of $\vec{u}$ in Proposition 3.5.1. In a similar way the second term in (3.7.4) admits the same upper bound which furthermore implies (3.7.3) for $m = 1$.

It remains to establish (3.7.3) for $m = 2$, in which case $Q_1(m)$ can be bounded by

$$\int_{\Sigma_\tau} w_\tau^{-1}\left(|\vec{u}||LLL\boldsymbol{u}| + |LL\vec{u}||L\boldsymbol{u}| + |L\vec{u}||LL\boldsymbol{u}|\right)|\partial_t LL\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} + \text{l.o.t},$$

where the lower order terms are those that show up when L acts on t^{-1} resulting

$$|L^\alpha(t^{-1})| \lesssim t^{-1} \tag{3.7.5}$$

for any m-tuple α. It suffices to bound the top order terms. Bounding L^∞-norm of $\vec{u}$ with the aid of Proposition 3.5.1 implies the first term can be bounded by

$$\begin{aligned}\int_{\Sigma_\tau} w_\tau^{-1}|\vec{u}||LLL\boldsymbol{u}||\partial_t LL\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-3/2}\mathfrak{E}_2(\tau)\int_{\Sigma_\tau} w_\tau^{-1/2}|LLL\boldsymbol{u}|w_\tau^{-1/2}|\partial_t LL\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau}\\ &\lesssim \tau^{-3/2}\mathfrak{E}_2(\tau)^3.\end{aligned}$$

The last two terms can be dealt with by using the interpolation Sobolev inequalities in Proposition 3.5.1. In particular, the second term can be bounded as

$$\begin{aligned}\int_{\Sigma_\tau} w_\tau^{-1}|LL\vec{u}||L\boldsymbol{u}||\partial_t LL\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-1}\int_{\Sigma_\tau} |LL\vec{u}|w_\tau^{1/3}|L\boldsymbol{u}|w_\tau^{1/6}|\partial_t LL\boldsymbol{u}|w_\tau^{-1/2}\,\mathrm{dvol}_{\Sigma_\tau}\\ &\lesssim \tau^{-1}\|\vec{u}\|_{\mathring{\mathcal{W}}_1^{2,3}}\|\boldsymbol{u}\|_{\mathring{\mathcal{W}}_1^{1,6}}\|\partial_t\boldsymbol{u}\|_{\mathring{\mathcal{W}}_{-1}^{2,2}}\\ &\lesssim \tau^{-1}\mathfrak{E}_1(\tau)\mathfrak{E}_2(\tau)^2.\end{aligned}$$

In a similar manner, the third term admits upper bound

$$\begin{aligned}\int_{\Sigma_\tau} w_\tau^{-1}|L\vec{u}||LL\boldsymbol{u}||\partial_t LL\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} &\lesssim \tau^{-1}\int_{\Sigma_\tau} |L\vec{u}|w_\tau^{1/3}|LL\boldsymbol{u}|w_\tau^{1/6}|\partial_t LL\boldsymbol{u}|w_\tau^{-1/2}\,\mathrm{dvol}_{\Sigma_\tau}\\ &\lesssim \tau^{-1}\mathfrak{E}_1(\tau)\mathfrak{E}_2(\tau)^2,\end{aligned}$$

which competes the proof of Claim (3.7.3).

The cubic terms can be dealt with similarly. In particular, the cubic terms in (3.7.1) by employing (3.7.2) can be decomposed into two parts, writing as $\mathcal{C}(m) = \mathcal{C}_1(m) + \mathcal{C}_2(m)$ with

$$\begin{aligned}\mathcal{C}_1(m) &= \int_{\Sigma_\tau} L^\alpha\left[(\vec{u})^2\cdot t^{-1}L^1\boldsymbol{u}\right]\cdot\partial_t L^\alpha\boldsymbol{u}\,\mathrm{dvol}_{\Sigma_\tau};\\ \mathcal{C}_2(m) &= \int_{\Sigma_\tau} L^\alpha\left[(\vec{u})^3 + \vec{u}\cdot\mathfrak{m}(\mathrm{d}\boldsymbol{u},\mathrm{d}\boldsymbol{u}) + (\vec{u})^2\cdot t^{-1}x^1\partial_t\boldsymbol{u}\right]\cdot\partial_t L^\alpha\boldsymbol{u}\,\mathrm{dvol}_{\Sigma_\tau}.\end{aligned}$$

for an m-tuple α with entries in $\{1,2,3\}$. Again the second term $\mathcal{C}_2(m)$ admit similar structure of cubic terms for the negative case and hence has the same bound as in Proposition

3.6.4. Here we recall L^i acting on $t^{-1}x^1$, or $w_\tau^{-1}x^1$ on Σ_τ, only contribute lower order terms by (2.1.8). The first item $\mathcal{C}_1$ can be dealt with by utilizing the extra decay of t^{-1}. We claim

$$|\mathcal{C}_1(m)| \lesssim \begin{cases} \tau^{-3}\mathfrak{E}_2(\tau)^2\mathfrak{E}_m(\tau)^2, & \text{if } m = 0,1; \\ \tau^{-2}\mathfrak{E}_2(\tau)^3\mathfrak{E}_1(\tau) + \tau^{-3}\mathfrak{E}_2(\tau)^4, & \text{if } m = 2. \end{cases} \tag{3.7.6}$$

For $m = 0$, the estimate above is a direct result of L^∞ bound of $\vec{u}$ in Proposition 3.5.1 and Hölder's inequality. For $m = 1$, utilizing (3.7.5) and $t = w_\tau$ on Σ_τ we need estimate

$$\int_{\Sigma_\tau} w_\tau^{-1}\left(|L\vec{u}||\vec{u}||L\boldsymbol{u}| + |\vec{u}|^2|LL\boldsymbol{u}| + |\vec{u}|^2|L\boldsymbol{u}|\right)|\partial_t L\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau}.$$

We bound the L^∞-norm of $L\boldsymbol{u}$ and $\vec{u}$ as in Proposition 3.5.1, apply $w_\tau^{-1} \le \tau^{-1}$ and distribute the weight appropriately, arriving at an upper bound:

$$\tau^{-3}\mathfrak{E}_2^2\int_{\Sigma_\tau}\left(w_\tau^{1/2}|L\vec{u}| + w_\tau^{-1/2}|LL\boldsymbol{u}| + w_\tau^{-1/2}|L\boldsymbol{u}|\right)w_\tau^{-1/2}|\partial_t L\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau}.$$

Applying Hölder's inequality implies the estimate for $m = 1$ in (3.7.6). Lastly, for $m = 2$ we need bound

$$\int_{\Sigma_\tau} w_\tau^{-1}\left(|LL\vec{u}||\vec{u}||L\boldsymbol{u}| + |L\vec{u}|^2|L\boldsymbol{u}| + |\vec{u}|^2|LLL\boldsymbol{u}|\right)|\partial_t LL\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} + \text{l.o.t.}$$

Utilizing L^∞ bound of $\vec{u}$ and $L\boldsymbol{u}$ in Proposition 3.5.1 and $w_\tau^{-1} \le \tau^{-1}$ yields an upper bound

$$\tau^{-2}\mathfrak{E}_2^2\int_{\Sigma_\tau}\left(w_\tau^{-1/2}|LL\boldsymbol{u}| + w_\tau^{1/2}|L\vec{u}| + \tau^{-1}w_\tau^{-1/2}|LLL\boldsymbol{u}|\right)w_\tau^{-1/2}|\partial_t LL\boldsymbol{u}|\,\mathrm{dvol}_{\Sigma_\tau} + \text{l.o.t.}$$

Then (3.7.6) for the case $m = 2$ follows directly by Hölder's inequality. $\square$

CHAPTER 4

THE MEMBRANE EQUATION

4.1 Introduction

Let M be a connected oriented $(d+1)$-manifold which is *immersed* in[1] $\mathbb{R}^{1,1+d}$ through the map $\Phi : M \to \mathbb{R}^{1,1+d}$. The immersion Φ is assumed to be *timelike*, in that the pullback metric $g \stackrel{\text{def}}{=} \Phi^*\mathfrak{m}$ on M is Lorentzian. A *relativistic membrane* shall refer to the image $\Phi(M)$ provided that any of the following equivalent conditions hold:

- the map Φ is a formal critical point to the volume functional

$$\mathcal{S}[\Psi] \stackrel{\text{def}}{=} \int_M \mathrm{dvol}(\Psi^*\mathfrak{m});$$

- the induced mean curvature vector H vanishes identically;
- the components of Φ satisfy the equation

$$\Box_g(x^\mu \circ \Phi) = 0,$$

  where $\Box_g$ is the Laplace-Beltrami operator on (M, g).

The assumption that Φ is an immersion implies that one can, locally, describe the membrane $\Phi(M)$ as a graph $\big(t, x^1, \ldots, x^d, \phi(t, x^1, \ldots, x^d)\big)$. In this coordinate system the equations of motion $\Box_g(x^\mu \circ \Phi) = 0$ take the divergence form

$$\sum_{\mu,\nu=0}^{d} \frac{\partial}{\partial x^\mu}\left(\frac{\mathfrak{m}^{\mu\nu}\partial_\nu\phi}{\sqrt{1+\mathfrak{m}(\mathrm{d}\phi,\mathrm{d}\phi)}}\right) = 0. \tag{4.1.1}$$

[1]We clarify that $\mathbb{R}^{1,1+d}$ is the Minkowski space with *one time dimension* and $(d+1)$-*space dimensions*.

This equation is variously known as the *membrane equation*, the *timelike minimal/maximal surface equation*, or the *Lorentzian vanishing mean curvature flow*. This is due to the interpretation that the graph of ϕ in $\mathbb{R}^{1,d} \times \mathbb{R} \cong \mathbb{R}^{1,1+d}$ is an embedded timelike hypersurface with zero mean curvature.

Solutions to (4.1.1) model extended test objects (world sheets), in the sense that the case where $d = 0$ reduces to the geodesic equation which models the motion of a test particle in $\mathbb{R}^{1,1}$. (The membrane equation can also be formulated with codimension greater than one; see [AAI06, Mil08].) The membranes can also interact with external forces which manifests as a prescription of the mean curvature; see [AC79, Hop13, Kib76, VS94] for some discussion of the physics surrounding such objects, and see [Jer11, Neu90] for rigorous justifications that membranes represent extended particles.

Our interest in the membrane equation arose, however, mainly due to it being an exceptional model of a quasilinear wave equation that is highly *non*-resonant. The exploration of resonant conditions in wave equations proceeded, historically, through two fronts. In the case of 1 spatial dimension, it has long been understood that hyperbolic systems with resonance (Lax's "genuinely nonlinear condition") lead to shock formation in finite time [Lax64, Lax73, Joh74]. For higher spatial dimensions, in the small-data regime, resonance has to compete with the dispersive decay enjoyed by wave equations. By now it is well understood that quasilinear wave equations enjoy small-data global existence in dimension $d \geq 4$, and also in dimensions $d = 2, 3$ when versions of Klainerman's null condition are satisfied [Kla80, Kla82, Kla84, Ali01a, Ali01b]. More recently the two fronts have met, where small-data shock formation for resonant quasilinear wave equations have been studied in spatial dimensions 2 and 3 [Ali01a, Ali01b, Chr07, Spe16, LS18]. For a recent review of the current understanding of small-data global existence versus shock formation in quasilinear waves, please see [HKSW16].

In a recent paper, Speck, Holzegel, Luk, and Wong studied the stability of plane-symmetric shock formation for quasilinear wave equations with resonance, under initial data perturbations that breaks the plane-symmetry [SHLW16]. More precisely, they start with a background simple-plane-symmetric solution to a quasilinear wave equation that is genuinely nonlinear, such that it forms a shock singularity in finite time. Such background solutions can be extracted from, for example, the late-time evolution of any small compactly supported initial data; we however allow our background solution to be of arbitrary "size". We were able to show that the shock formation is stable under arbitrary initial data perturbations that breaks the simple-plane-symmetry, provided that the perturbation is small compared to the background solution.

A natural follow-up question is: *when genuine nonlinearity fails, in particular when there exists simple-plane-symmetric global solutions to the quasilinear wave equation, is the global existence stable under small, symmetry-breaking initial data perturbations?*

Returning to the membrane equation, we note that the equation is highly non-resonant. It satisfies a stronger null condition than is typical of quasilinear waves in 2 or 3 dimensions. This was explicitly exploited to show global well-posedness of the small-data problem first by Brendle [Bre02] when $d = 3$ and then by Lindblad [Lin04] in $d = 2$ *and* $d = 1$. The $d = 1$ case is surprising as, there being no dispersive decay for the one-dimensional wave, any resonance, even arbitrarily high order, can lead to finite-time blow-up. Wong explored this case in more detail geometrically [Won17b] and enlarged the class of initial data for which global existence holds.

Our focus on the membrane equation in this chapter then is due to the fact that (i) as a consequence of [Lin04] and [Won17b], there exists robust families of global plane-symmetric solutions to the membrane equation, and (ii) the null geometry of such solutions are well understood by the analyses of [Won17b]. We remark that, while not explicitly stated, following the same method of proof of the main theorem in [Won17b], one can show that the global simple planewave solutions described below in Section 4.2.1 are automatically stable under plane-symmetric perturbations that are *not necessarily simple*.

We state and prove our main result in dimension $d = 3$; as described in the previous paragraph, the result is effectively known in $d = 1$. Our proof also works in all dimensions $d \geq 3$ thanks to the improved dispersive decay of solutions to the linear wave equation in higher spatial dimensions. Our proof however doesn't work in $d = 2$ due to certain technical losses of decay (see Remark 4.5.1 below). In [LZ19] the authors were able to prove a similar result in $d = 2$ with weaker asymptotic control; see Remark 4.1.4 for further discussion.

One should note, at this juncture, that the non-resonance of the membrane equation is only effective at preventing a certain type of singularity formation. Indeed, far away from the nearly-simple-planewave regime that we consider in the present chapter, singularities are known to arise from regular initial data. In the case where $d = 1$ these were analyzed by Nguyen and Tian [NT13] and Jerrard, Novaga, and Orlandi [JNO15]; while their analyses concentrate on the case with spatially periodic domain, by finite speed of propagation the same singularity formation can be localized and placed in our context. Analogues of [NT13, JNO15] in higher spatial dimensional backgrounds were studied by Wong in [Won18b]. In these cases the singularities are *not* of shock-type, but rather appear due to the degeneration of the principal symbol of the evolution.

4.1.1 Our main result and discussions

The answer to the question asked in the previous section is in the affirmative: we show that simple-planewave solutions to the membrane equation are stable under small initial data perturbations. The precise version of our main theorem is Theorem 4.5.8; there we state the result as a small-data global existence result for the corresponding perturbation equations, after a nonlinear change of independent variables that corresponds to a gauge choice. Here we state a slightly less precise version in terms of the original variables.

Theorem 4.1.1. *Fix the dimension $d = 3$. Let Υ denote a smooth simple-plane-symmetric solution to* (4.1.1) *with finite extent in its direction of travel. Fix a bounded set $\Omega \subset \mathbb{R}^3$. There exists some $\epsilon_0 > 0$ depending on the background Υ and the domain Ω, such that for* *any* $(\psi_0, \psi_1) \in (H^5(\mathbb{R}^3) \cap C_0^\infty(\Omega)) \times (H^4(\mathbb{R}^3) \cap C_0^\infty(\Omega))$ *with* $\|(\psi_0, \psi_1)\| < \epsilon_0$, *the initial value problem to* (4.1.1) *with initial data*

$$\phi(0,x) = \Upsilon(0,x) + \psi_0(x), \qquad \partial_t \phi(0,x) = \partial_t \Upsilon(0,x) + \psi_1(x)$$

has a global solution that converges in $C^2(\mathbb{R}^3)$ *to* Υ *as* $t \to \pm\infty$.

Remark 4.1.2 (Finite extent in the direction of travel)**.** We ask that Υ essentially represent a travelling "pulse". For example, taking plane-symmetry to mean constant in the x^2 and x^3 variables, Υ would be a function of $t = x^0$ and x^1 alone. We ask that for any fixed t the function Υ vanishes for all sufficiently large x^1. We make heavy use of this finite extent property in the course of the proof (see Lemma 4.3.13).

Remark 4.1.3 (Simplicity)**.** By a simple planewave solution we refer to a solution that is not only constant in the x^2 and x^3 variables, but one such that the differential $\mathrm{d}\Upsilon$ is null with respect to the dynamic metric. In other words, a simple planewave solution is one that propagates along only one (and not both) of the characteristic directions of the nonlinear wave equation.

The assumption of simplicity is only to keep the argument simple (pun intended). In fact, assuming finite extent of the initial data for the plane-symmetric background, automatically by the sharp Huygen's principle for one dimensional waves, after a finite-length of time the background will decouple into two spatially disjoint simple planewaves travelling in opposite directions. By Cauchy stability of the finite-time initial value problem, we see that the theorem for the simple planewave background also implies the theorem for general, globally existing plane-symmetric backgrounds such as those demonstrated to exist in [Lin04, Won17b].

We note here, however, that another feature of simplicity is that simple-planewave solutions exist for arbitrary pulse profile (see Section 4.2.1 below). The same is not the case for non-simple planewave solutions: large interacting waves can form finite-time singularities.

Remark 4.1.4 (Dimensionality)**.** The theorem above is stated for $d = 3$. The same arguments can be used to prove stability for all dimensions $d \geq 3$ (in fact the arguments can be significantly further simplified when $d \geq 5$). One needs to modify the degree of regularity required. When $d = 3$ the data is taken to be small in $H^k \times H^{k-1}$ with $k = 5$. When $d \geq 4$ is even we will need $k = d + 3$, and when $d \geq 5$ is odd we will need $k = d + 2$. Compare to the discussion in Section 4.4 below.

As mentioned before in this introduction, the $d = 1$ analogue of the result essentially follows from the arguments in [Won17b]. This leaves the case $d = 2$, which received attention from Liu and Zhou [LZ19]. Aside from minor technical differences in how we approach the energy and pointwise estimates, a difference appears in how we linearize around the background solution. In the present chapter we use the geometric normal graphical gauge (see below) adapted to the background traveling wave, while in [LZ19] they used the gauge adapted to the trivial solution. Our gauge has the advantage that the perturbation equations contain no *linear* potential from the background; the price paid being the appearance of nonlinear contributions of lower order whose null structure are less apparent. In [LZ19], they were so far able to show global existence for the perturbation equations but only C^0 (and *not* C^1) convergence to the background. The lack of higher-derivative convergence can be attributed, at least in part, to their gauge choice. Based on our own work we have high hopes that in fact C^2 convergence can be proven to hold, though at present there are some technical difficulties for even showing global existence using a direct extension of our method; see also Remark 4.5.1.

Our main theorem is not a straight-forward small-data global existence result for a quasilinear wave equation. The equations satisfied by the perturbations around *large* solutions generally include coefficients contributed by background, the effects of which must also be captured. In our problem, to leading order the perturbation equation looks like

$$\Box_{\mathfrak{m}}\phi + \phi\Upsilon''(\partial_t + \partial_{x^1})^2\phi + \Upsilon''(\partial_t\phi + \partial_{x^1}\phi)^2 = 0. \tag{4.1.2}$$

Here $\Box_{\mathfrak{m}}$ is the flat wave operator, and the background pulse is assumed to be travelling in the $+x^1$ direction, so has compact support in the $(t - x^1)$ variable.

The first thing to notice is that the linearized equation is the linear wave equation on Minkowski space. This is a special geometric feature of simple-travelling wave solutions to the membrane equation. To expose this special linear structure, one needs to make an appropriate gauge choice involving a nonlinear change of variables adapted to the background Υ, which essentially re-writes our perturbation equations as a graph in the normal bundle of Υ, interpreted as a submanifold of $\mathbb{R}^{1,1+d}$. It is well-known that the membrane equation has good structure in such "normal graphical gauge": in this formulation the linearized equation can be expressed as the geometric wave operator adapted to the induced Lorentzian metric on the background Υ, plus possibly a potential term. This gauge was also used, for example, in [DKSW16].

In view of this special geometric feature, we do not need to develop special methods to perform the linear analysis. On the other hand, the function Υ'' is *non-decaying* and has support within the "wave zone"; this significantly complicates the analysis of the nonlinear terms, especially since these nonlinearities are not in the shape of classical null forms. This is in contrast with the analyses in [DKSW16] where the stability of another "large data" solution to the membrane equation was considered. The background solution in that case is the static catenoid solution. The nontrivial catenoid background introduced a low-frequency correction to the linear evolution (in fact giving an exponentially growing mode). But as the background is asymptotically flat, the high-frequency evolution, especially in the wave-zone where it is the most delicate when it comes to the nonlinear interactions, is entirely captured by classical null structures. In particular the nonlinearities do not introduce new difficulties beyond the adjustments made for the modified linear evolution. Another difference with our work and [DKSW16] is that they prove that the catenoid is globally stable under *axially symmetric* codimension one initial perturbations, whereas we prove that our planewave solution is globally stable under an open set of *symmetry breaking* perturbations. Their symmetry assumptions on the perturbations are there to avoid the issue of trapped geodesics on the catenoid.

In the present chapter, on the other hand, the focus is entirely on the nonlinearity, with the main difficulty arising precisely from the non-decaying background Υ''. At this point it may be worth drawing comparison to another large-data (semi-)global existence result for the membrane equation. In [WW17], the authors studied the membrane equation with initial data given as a small perturbation of an out-going "short-pulse". The (semi-)global existence (note that by their choice of initial data, the result in [WW17] is not time-symmetric!) mechanism in this case is essentially still the classical null condition of Klainerman. The strong non-resonance condition of the membrane equation means that the "large" short-pulse background does not interact with itself; and in fact the pulse itself *decays* like the solution to the linear wave equation. Putting this together with the fact that the nonlinearities in (4.1.1) are cubic, this means that heuristically we can understand the result of [WW17] as very similar to the large data stability result for the wave maps system proven in [Sid89], which also required the "background geodesic solution" to be one with finite (weighted) energy, and hence decays like finite energy solutions to the linear wave equation. These types of systems can be modeled by the quasilinear system

$$\Box_{\mathfrak{m}}\psi_1 = 0,$$
$$\Box_{\mathfrak{m}}\psi_2 = \mathfrak{m}(\nabla\psi_1, \nabla\mathfrak{m}(\nabla\psi_2, \nabla\psi_2)) + \mathfrak{m}(\nabla\psi_2, \nabla\mathfrak{m}(\nabla\psi_1, \nabla\psi_2)).$$

Even when ψ_1 is a "large" solution, it contributes enough decay that the nonlinearities for the second equation decay at an integrable rate. Together with the fact that the nonlinearity is quadratic in ψ_2, we can upgrade the smallness and close the bootstrap. Note that the decay of ψ_1 is crucial, as, in the second term of the nonlinearity we see components like

$$(\partial_t + \partial_r)^2\psi_1 \cdot (\partial_t\psi_2 - \partial_r\psi_2)^2.$$

This is a resonant interaction in ψ_2, whose contribution is significantly ameliorated by the fact that $(\partial_t + \partial_r)^2\psi_1$ should decay like $t^{-3/2}$ (or better) in $\mathbb{R}^{1,3}$ or t^{-1} (or better) in $\mathbb{R}^{1,2}$. If we were to replace the ψ_1 factor by a generic bounded function in $\mathbb{R}^{1,3}$ (or a function decaying no faster than $1/\sqrt{t}$ in $\mathbb{R}^{1,2}$) this term will lead to finite-time blow-up.

Returning to our equation (4.1.2), we see that we have precisely this type of resonant interaction with a non-decaying coefficient. Instead of coefficient decay, we need to exploit a different aspect of the null structure of the original membrane equation (4.1.1). What we will use is the fact that Υ'' has compact support in the $(t - x^1)$ variable. The resonant interacting terms $(\partial_t\phi + \partial_{x^1}\phi)$ represent waves traveling in directions transverse[2] to the level sets of $t - x^1$. In particular, we expect that the resonant interaction to only take place for a bounded length of time (for each wave packet). Our main mechanism would therefore be something similar to that which drives Shatah's space-time resonance arguments [Sha10], but captured in a purely physical space manner.

Of course, we have to pay a price for this non-decay. This manifests in us having to use a polynomially-growing energy hierarchy when using the vector field method. In fact, our higher order energies, starting with the second (controlling the third derivatives in L^2), will grow in time, with each additional order differentiation growing one order faster in time. One should compare to classical applications of the vector field method where all but the top-order energies are bounded in time, with the top-order typically exhibiting no worse than a log growth. The upshot of this energy hierarchy is that we lose strong peeling properties of the solutions. (See Remark 4.5.9.)

4.1.2 Outline of the chapter

The remainder of this chapter is organized as follows: we first discuss the background planewave solutions Υ. These solutions are introduced in Section 4.2.1. Their basic ge-

[2]In $(1+1)$-dimensions, the linear wave equation can be expressed as $(\partial_t - \partial_{x^1})(\partial_t\varphi + \partial_{x^1}\varphi) = 0$, and since $(\partial_t - \partial_{x^1})(t - x^1) = 2$, one sees that $\partial_t\varphi + \partial_{x^1}\varphi$ is a bonafide traveling wave transverse to the level sets of $t - x^1$.

ometric properties and our gauge choice for studying the perturbations are described in Section 4.2.2.

We next discuss the basic analytic tools used in our arguments; in Section 4.3.1 we recall the global Sobolev inequalities of Chapter 2 adapted to the geometry of the planewave background, in Section 4.3.2 we develop a weighted vector field algebra to help simplify our analyses of the nonlinear terms using more schematic notations.

In Section 4.4 we study the semilinear model problem $\Box_{\mathfrak{m}}\phi = \Upsilon''(\partial_{\underline{u}}\phi)^2$, obtained from dropping the quasilinearity from (4.1.2). This model problem turns out to capture already the majority of the difficulty one faces when analyzing the full problem. We prove small-data global wellposedness for the semilinear model in all dimensions ≥ 3. There are certain additional technical difficulties for studying the quasilinear model (4.1.2) in $d = 2$ due to the fact one expects even the *first order energy* exhibits polynomial growth there, and the loss seems too strong to overcome with the methods described in this chapter; therefore we also omit a detailed treatment of the $d = 2$ semilinear model.

The remainder of the chapter is devoted to studying the full quasilinear problem in $d = 3$, and stating and proving a more precise version of Theorem 4.1.1. In Section 4.5 we perform first some preliminary computations casting the equations for the perturbation ϕ and its higher order derivatives in schematic form to prepare for analysis. As many of the computations are long and involved, we delegate sketches of the arguments separately to the Appendix. At the end of the section we state our Main Theorem 4.5.8. As usual, we will prove our Main Theorem by a bootstrap argument for our energy hierarchy. In Section 4.6 we define our energy quantities, outline our main energy estimate, state our bootstrap assumptions, and derive some immediate consequences that do not involve the equations of motion. Section 4.7 is devoted to proving *a priori* estimates for our equations of motion, based on the bootstrap assumptions. These are combined in Section 4.8 to show that the bootstrap assumptions can be improved, and thereby hold for all time and global existence follows.

4.2 The background solution

In this section we first exhibit simple planewave solutions to the membrane equation, and describe their geometry. These solutions are traveling waves and exist for all time; our goal is to analyze their stability under small non-plane-symmetric perturbations. To do so we recast the stability problem as a small-data Cauchy problem for the perturbation. In the second part of this section we exploit the geometric interpretation of the solutions as minimal submanifolds of higher dimensional Minkowski space to make a convenient choice of gauge, and derive the corresponding perturbation equations. The gauge choice allows us to simplify the analysis of the linearized dynamics. As the membrane equation itself is a quasilinear wave equation, when linearizing around a fixed nontrivial background solution, typically the background contributes to the linearized dynamics (e.g. in [DKSW16] where the background contributes a potential term leading to generic instability of the system). For the membrane equation in Minkowski space, however, it is known [CB76] that the potential term in the linearized dynamics for perturbations parametrized by the normal bundle is given by the double contraction of the extrinsic curvature of the embedding of the background solution. For simple planewaves, this potential term vanishes [Won17b]. The gauge choice below makes this explicit and shows that the perturbed system can be described by a *quasilinear perturbation* of the linear wave equation on Minkowski space, with the background solution only appearing as *coefficients of the nonlinearity*.

4.2.1 Simple planewave solutions to the membrane equation

Let $\Upsilon \in C^\infty(\mathbb{R};\mathbb{R})$ be arbitrary. One easily sees that the function $\mathring{\phi} : \mathbb{R}^{1+d} \to \mathbb{R}$ defined by[3]

$$\mathring{\phi}(t, x^1, x^2, \dots, x^d) = \Upsilon(t + x^1) \tag{4.2.1}$$

solves (4.1.1), seeing as $\mathrm{d}\mathring{\phi}(t, x^1, \dots) = \Upsilon'(t + x^1)\, \mathrm{d}(t + x^1)$ and hence $\mathfrak{m}(\mathrm{d}\mathring{\phi}, \mathrm{d}\mathring{\phi}) \equiv 0$ and $\mathfrak{m}^{\mu\nu}\partial^2_{\mu\nu}\mathring{\phi} \equiv 0$. The simple planewave background will be interpreted as the graph of $\mathring{\phi}$ in $\mathbb{R}^{1,d+1}$, the $(d+2)$-dimensional Minkowski space equipped with the standard Minkowski metric M. That is to say, we consider the embedding $\mathbb{R}^{1+d} \hookrightarrow \mathbb{R}^{1,d+1}$, given by

$$(t, x^1, \dots, x^d) \mapsto (t, x^1, \dots, x^d, \mathring{\phi}(t, x))$$

with the first component fixed as the timelike one. The $\mathfrak{m}(\mathrm{d}\phi, \mathrm{d}\phi) \neq -1$ implies that the induced metric on the graph of ϕ is Lorentzian and non-degenerate. By the analysis of [Won17b] this induced metric is *flat*; this fact can also be seen through the following explicit computations.

Denoting the above embedding by Φ, the induced metric can be in fact given by the line element

$$\Phi^* M = \mathrm{d}s^2 = (-1 + (\partial_t \mathring{\phi})^2)\, \mathrm{d}t^2 + 2\partial_t \mathring{\phi} \partial_{x^1} \mathring{\phi}\, \mathrm{d}t\, \mathrm{d}x^1 + (1 + (\partial_{x^1} \mathring{\phi})^2)\, \mathrm{d}(x^1)^2 \\ + \mathrm{d}(x^2)^2 + \dots + \mathrm{d}(x^d)^2.$$

Using that $\partial_t \phi(t, x) = \partial_{x^1} \phi(t, x) = \Upsilon'(t + x^1)$, we see that if we define

$$\begin{cases} u \stackrel{\text{def}}{=} t + x^1, \\ \underline{u} \stackrel{\text{def}}{=} \frac{1}{2}\left[t - x^1 - \int_0^{x^1 + t} (\Upsilon')^2(\tau)\, \mathrm{d}\tau\right]; \end{cases} \tag{4.2.2}$$

[3]We've made the choice to have our background travelling waves move "to the left", i.e. as a function of $t + x^1$. Note that for the analyses in [SHLW16] the simple waves move "to the right". We beg those readers familiar with the previous work to indulge us and mentally reorient the space-time and relabel the function u as needed.

that the line element can be alternatively written as the Minkowski metric in standard double-null form

$$\mathfrak{m} = \mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}\underline{u} + \mathrm{d}(x^2)^2 + \cdots + \mathrm{d}(x^d)^2. \tag{4.2.3}$$

The functions u and $\underline{u}$ solve the eikonal equation $\mathfrak{m}(\nabla u, \nabla u) = \mathfrak{m}(\nabla \underline{u}, \nabla \underline{u}) = 0$. For the subsequent analyses we will parametrize using the coordinates $\{u, \underline{u}, x^2, \ldots, x^d\}$.

Remark 4.2.1. Note that there are two Minkowski metrics involved in the construction: (1) The metric on the ambient space $\mathbb{R}^{1,d+1}$, which is denoted by M. (2) The induced Minkowski metric on the planewave background given by double-null coordinates $(u, \underline{u}, \hat{x}) \in \mathbb{R}^{1,d}$, denoted by $\mathfrak{m}$.

For completeness, we note that the change of variables can be inverted:

$$\begin{cases} t = \frac{1}{2}u + \underline{u} + \frac{1}{2}\int_0^u (\Upsilon')^2(\tau)\,\mathrm{d}\tau, \\ x^1 = \frac{1}{2}u - \underline{u} - \frac{1}{2}\int_0^u (\Upsilon')^2(\tau)\,\mathrm{d}\tau. \end{cases} \tag{4.2.4}$$

For convenience, we note that relative to this coordinate system, our simple planewave solution is given by the embedding

$$(u, \underline{u}, \hat{x}) \mapsto (t, x^1, \hat{x}, \Upsilon(u)) \in \mathbb{R}^{1,d+1} \tag{4.2.5}$$

where t and x^1 are given as functions of $u, \underline{u}$ by (4.2.4), and for convenience we denote by $\hat{x} = (x^2, \ldots, x^d)$.

We finish this subsection by computing the extrinsic curvature (second fundamental form) of the embedding (4.2.5). The change of variables (4.2.4) implies that the vector fields

$$\partial_{\underline{u}} = (1, -1, 0, \ldots, 0), \qquad \partial_u = \Big(\frac{1}{2}(1 + \Upsilon'(u)^2), \frac{1}{2}(1 - \Upsilon'(u)^2), 0, \ldots, 0, \Upsilon'(u)\Big).$$

Denote by $n : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d+1}$ the unit normal vector field (with respect to the Minkowski metric on $\mathbb{R}^{1,d+1}$) of the embedding (4.2.5) given by

$$n(u, \underline{u}, \hat{x}) = (-1)^{d-1}(-\Upsilon'(u), \Upsilon'(u), 0, \ldots, 0, -1). \tag{4.2.6}$$

The expression (4.2.6) can be computed from

$$n^{\alpha} = (M^{-1})^{\alpha\kappa}\epsilon_{\kappa,\beta,\gamma,\sigma_2,\ldots,\sigma_{d-1}}(\partial_u)^{\beta}(\partial_{\underline{u}})^{\gamma}(\partial_{x^2})^{\sigma_2}\cdots(\partial_{x^{d-1}})^{\sigma_{d-1}},$$

where $\alpha,\kappa,\beta,\gamma,\sigma_2,\ldots,\sigma_{d-1} \in \{0,1,\ldots,d-1\}$ and $\epsilon_{\kappa,\beta,\gamma,\sigma_2,\ldots,\sigma_{d+1}}$ is the anti-symmetric symbol normalized by $\epsilon_{0,1,2,\ldots,d-1} = 1$. The second fundamental form can then be computed to equal

$$II = (-1)^{d-1}\Upsilon''(u)\,\mathrm{d}u^2. \tag{4.2.7}$$

(We use the convention $II(\partial_u,\partial_u) = \langle \partial_u n, \partial_u\rangle_M$.) Notice that II is indeed trace-free with respect to the induced metric as a consequence of and additionally the double contraction $II : II$ with respect to the induced metric also vanishes, both a consequence of the eikonal equation.

4.2.2 The gauge choice and the perturbed system

Small perturbations of the embedding (4.2.5) reside within a tubular neighborhood of the background. We parametrize the perturbations as a graph within the normal bundle, analogously to the analysis in [DKSW16]; that is, we look for embeddings of the form

$$(u,\underline{u},\hat{x}) \mapsto (t,x^1,\hat{x},\Upsilon(u)) + \phi(u,\underline{u},\hat{x})\cdot n(u,\underline{u},\hat{x}) \tag{4.2.8}$$

where $\phi : \mathbb{R}^{1+d} \to \mathbb{R}$ is the *height* of the graph, and n is the unit normal as defined in (4.2.6). The induced metric for this perturbation will be denoted by g; it is given by the pull-back of the Minkowski metric M on $\mathbb{R}^{1,d+1}$ by the embedding (4.2.8)

$$g = \mathfrak{m} + \mathrm{d}\phi\otimes\mathrm{d}\phi - 2\phi\Upsilon''\,\mathrm{d}u\otimes\mathrm{d}u. \tag{4.2.9}$$

Its corresponding volume element can be computed to be

$$\mathrm{dvol}_g = \sqrt{|g|}\,\mathrm{d}u\mathrm{d}\underline{u}\mathrm{d}\hat{x}$$

where

$$|g| \stackrel{\text{def}}{=} 1 + \mathfrak{m}(\nabla\phi,\nabla\phi) + 2\phi\Upsilon''(\partial_{\underline{u}}\phi)^2. \tag{4.2.10}$$

We note that g is a perturbation of the Minkowski metric $\mathfrak{m}$ with terms both quadratic and linear in ϕ. For later computations it is helpful to also record the perturbations truncated to the linear terms, which we will denote by $\mathring{g}$

$$\mathring{g} \stackrel{\text{def}}{=} \mathfrak{m} - 2\phi\Upsilon'' \,\mathrm{d}u \otimes \mathrm{d}u. \tag{4.2.11}$$

The inverses of g and $\mathring{g}$ can be computed explicitly in the double null (relative to $\mathfrak{m}$) coordinates $(u, \underline{u}, \hat{x})$. For $\mathring{g}$ one finds

$$\mathring{g}^{-1} = \mathfrak{m}^{-1} + 2\phi\Upsilon'' \,\partial_{\underline{u}} \otimes \partial_{\underline{u}}. \tag{4.2.12}$$

Note that this implies

$$|g| = 1 + \mathring{g}^{-1}(\nabla\phi, \nabla\phi). \tag{4.2.13}$$

Using that $g = \mathring{g} + \mathrm{d}\phi \otimes \mathrm{d}\phi$, we can apply the Sherman-Morrison formula [SM50] to obtain

$$g^{-1} = \mathring{g}^{-1} - \frac{1}{|g|}\Big(\mathring{g}^{-1} \cdot \partial\phi\Big) \otimes \Big(\mathring{g}^{-1} \cdot \partial\phi\Big). \tag{4.2.14}$$

Notation 4.2.2 (Index raising and lowering)**.** In the computations to follow, one frequently needs to lower or raise indices with respect to any of g / g^{-1}, $\mathring{g}$ / $\mathring{g}^{-1}$, or $\mathfrak{m}$ / $\mathfrak{m}^{-1}$. We will adopt the following conventions

- The unadorned musical operators $\flat$ / $\sharp$ are used for lowering and raising indices with respect to the Minkowski metric $\mathfrak{m}$ of the background simple-planewave solution.

- Implicitly lowered / raised indices are always with the Minkowski metric $\mathfrak{m}$, so $\partial^j\phi$ refers to $\mathfrak{m}^{jk}\partial_k\phi$.

- When it is clear from the context, we will sometimes omit the index -1 denoting inverses for brevity. For example, we write $\mathfrak{m}(\nabla\phi, \nabla\phi)$ instead of $\mathfrak{m}^{-1}(\nabla\phi, \nabla\phi)$ since $\nabla\phi$ are naturally covariant and so we will need the contravariant metric $\mathfrak{m}^{-1}$. Similarly, if we write $g^{\mu\nu}\partial_\nu\phi$ it should be interpreted as $(g^{-1})^{\mu\nu}\partial_\nu\phi$.

- Index manipulations with respect to the dynamical metrics g and $\mathring{g}$ will always be adorned. So for example we will write

$$\partial^{\mathring{g}\sharp}\phi = \mathring{g}^{-1}\cdot\partial\phi, \qquad \partial^{g\sharp}\phi = g^{-1}\cdot\partial\phi$$

with corresponding index notation

$$\partial^{\mathring{g}\sharp j}\phi = \mathring{g}^{jk}\partial_k\phi, \qquad \partial^{g\sharp j}\phi = g^{jk}\partial_k\phi.$$

With the notation announced above, we can equivalently write

$$g^{-1} = \mathring{g}^{-1} - \frac{1}{|g|}\partial^{\mathring{g}\sharp}\phi\otimes\partial^{\mathring{g}\sharp}\phi.$$

For the embedding (4.2.8) to have vanishing mean curvature (i.e. satisfy the membrane equation), it must be a formal stationary point of the volume functional $\phi\mapsto\int \mathrm{dvol}_g$. The perturbation equations satisfied by ϕ can be derived as the corresponding Euler-Lagrange equations, as shown below.

Denoting by $\mathcal{L} = \sqrt{|g|} = \sqrt{1+\mathfrak{m}(\nabla\phi,\nabla\phi)+2\phi\Upsilon''(\partial_{\underline{u}}\phi)^2}$ the Lagrangian density, the corresponding Euler-Lagrange equation is

$$\frac{\delta\mathcal{L}}{\delta\phi} = \frac{\partial}{\partial u}\left(\frac{\delta\mathcal{L}}{\delta\phi_u}\right) + \frac{\partial}{\partial\underline{u}}\left(\frac{\delta\mathcal{L}}{\delta\phi_{\underline{u}}}\right) + \frac{\partial}{\partial\hat{x}}\left(\frac{\delta\mathcal{L}}{\delta\phi_{\hat{x}}}\right) \tag{4.2.15}$$

where we use the subscript on ϕ to denote partial differentiation. Expanding $\mathfrak{m}(\nabla\phi,\nabla\phi) = -2\partial_u\phi\partial_{\underline{u}}\phi + (\partial_{\hat{x}}\phi)^2$ we compute

$$\frac{\delta\mathcal{L}}{\delta\phi} = \mathcal{L}^{-1}\Upsilon''(\phi_{\underline{u}})^2 \qquad\qquad \frac{\delta\mathcal{L}}{\delta\phi_{\hat{x}}} = \mathcal{L}^{-1}\phi_{\hat{x}}$$

$$\frac{\delta\mathcal{L}}{\delta\phi_u} = \mathcal{L}^{-1}(-\phi_{\underline{u}}) \qquad\qquad \frac{\delta\mathcal{L}}{\delta\phi_{\underline{u}}} = \mathcal{L}^{-1}(-\phi_u + 2\Upsilon''\phi\phi_{\underline{u}}).$$

So the Euler-Lagrange equation reads

$$\partial_\mu\left(\frac{\mathring{g}^{\mu\nu}\partial_\nu\phi}{\mathcal{L}}\right) = \mathcal{L}^{-1}\Upsilon''(\phi_{\underline{u}})^2. \tag{4.2.16}$$

Observe that by (4.2.14) we have

$$\partial^{g\sharp}\phi = \frac{1}{|g|}\partial^{\mathring{g}\sharp}\phi.$$

This implies that we can rewrite (4.2.16) as

$$\Box_g \phi = |g|^{-1} \Upsilon''(\phi_{\underline{u}})^2; \tag{4.2.17}$$

here $\Box_g$ refers to the Laplace-Beltrami operator of the metric g, given in local coordinates by

$$\Box_g f = \frac{1}{\sqrt{|g|}} \partial_\mu \Big(\sqrt{|g|} g^{\mu\nu} \partial_\nu f \Big).$$

As the metric g depends on the first jet of the unknown ϕ, the principal part of the (4.2.17) may be different from $g^{\mu\nu}\partial^2_{\mu\nu}\phi$. For our equation, this turns out not to be an issue, as can be seen when we take the first coordinate partial derivatives of (4.2.16). With the aid of the relation (4.2.14) between g^{-1} and $\mathring{g}^{-1}$ we obtain

$$\partial_\lambda \partial_\mu \left(\frac{\partial \mathring{g}^{\sharp\mu} \phi}{\mathcal{L}} \right) = \partial_\mu \left(\frac{\partial g^{\sharp\mu} \partial_\lambda \phi}{\sqrt{|g|}} \right) + \partial_\mu \left(\frac{\partial_\lambda \mathring{g}^{\mu\nu} \partial_\nu \phi}{\sqrt{|g|}} - \frac{1}{2} \frac{\partial \mathring{g}^{\sharp\mu}\phi}{|g|^{3/2}} \partial_\lambda \mathring{g}^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \right).$$

Noticing that the derivatives $\partial_\lambda \mathring{g}$ depends only on the *first* derivatives of ϕ, and not the second, we see that the principal term are all captured in the first term on the right in the above identity.

We can simplify the identity further. Notice that

$$\partial_\lambda \mathring{g}^{-1} = \partial_\lambda (2\phi\Upsilon'')\, \partial_{\underline{u}} \otimes \partial_{\underline{u}}$$

this implies

$$\begin{aligned}\partial_\mu \left(\frac{\partial_\lambda \mathring{g}^{\mu\nu} \partial_\nu \phi}{\sqrt{|g|}} - \frac{1}{2} \frac{\partial \mathring{g}^{\sharp\mu}\phi}{|g|^{3/2}} \partial_\lambda \mathring{g}^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \right) &= 2\partial_{\underline{u}} \left(\frac{\partial_\lambda (\phi\Upsilon'') \partial_{\underline{u}} \phi}{\sqrt{|g|}} \right) \\ &- \underbrace{\partial_\mu \left(\frac{\partial \mathring{g}^{\sharp\mu}\phi}{|g|^{1/2}} \right)}_{|g|^{-1/2}\Upsilon''(\partial_{\underline{u}}\phi)^2} \frac{1}{|g|} \partial_\lambda (\phi\Upsilon'') (\partial_{\underline{u}}\phi)^2 - \frac{\partial \mathring{g}^{\sharp\mu}\phi}{|g|^{1/2}} \partial_\mu \left(\frac{1}{|g|} \partial_\lambda (\phi\Upsilon'') (\partial_{\underline{u}}\phi)^2 \right).\end{aligned}$$

So we conclude

$$|g|^{-\frac{1}{2}-\frac{2}{d-1}}\Box_{\tilde{g}}\partial_\lambda\phi = \partial_\lambda\left(|g|^{-1/2}\Upsilon''(\phi_{\underline{u}})^2\right) - 2\partial_{\underline{u}}\left(\frac{\partial_\lambda(\phi\Upsilon'')\partial_{\underline{u}}\phi}{\sqrt{|g|}}\right) + |g|^{-3/2}\Upsilon''\partial_\lambda(\phi\Upsilon'')(\partial_{\underline{u}}\phi)^4 + |g|^{-1/2}\partial\mathring{g}^{\sharp\mu}\phi\partial_\mu\left(\frac{1}{|g|}\partial_\lambda(\phi\Upsilon'')(\partial_{\underline{u}}\phi)^2\right), \quad (4.2.18)$$

where we have introduced the conformal metric

$$\tilde{g} = |g|^{-\frac{2}{d-1}}\cdot g \quad (4.2.19)$$

where d is, recall, the number of spatial dimensions. The conformal metric $\tilde{g}$ has its Laplace-Beltrami operator as

$$\Box_{\tilde{g}} f = |g|^{\frac{1}{2}+\frac{2}{d-1}}\partial_\mu\left(\frac{1}{\sqrt{|g|}}g^{\mu\nu}\partial_\nu f\right)$$

which has the same principal part as $\Box_g$.

Remark 4.2.3. Observe that (4.2.17) and (4.2.18) are *geometric* quasilinear wave equations that linearize to the linear wave equation on $\mathbb{R}^{1,d}$. The *quadratic nonlinearities* include, as can be seen, the *resonant* semilinear interaction $(\partial_{\underline{u}}\phi)^2$ as well as the *weakly resonant* quasilinear interaction $\phi(\partial^2_{\underline{uu}}\phi)$.

That we will be able to prove global existence for this equation (and not suffer from shock formation in finite time) is due to the background Υ'' which accompanies the apperance of such resonant terms, and localizes the resonant interactions to the region $t \approx -x^1$; one can think of Υ'' as $\partial^2_{uu}\Upsilon$, exposing the null condition that was present in the original membrane equation (4.1.1). However, as the background function Υ has non-compact (in the $\hat{x}$ direction) support, and is non-decaying (in time), the improved decay we obtain due to this space-time localization is weaker than in classical studies of nonlinear waves with null condition. Such issues and their ramifications are discussed in more detail in Section 4.4 where we examine a semilinear model that captures the main analytical difficulties.

4.3 Preliminary L^2 analysis

We will analyze (4.2.17) using the vector field method (VFM) adapted to hyperboloidal foliations. As we saw in the analysis of the wave maps equation in Chapter 3, this variant of the VFM allows us to derive a priori estimates using only the ∂_t-multiplier (see the discussion following (2.2.8)) and not the Morawetz K multiplier (see Remark 2.2.4). To efficiently handle the coefficients Υ'' present in (4.2.17) using *only* the Lorentz boosts as commutators, we will develop in the second part of this section a weighted vector field algebra. The combination of these techniques will be first illustrated in a model semilinear problem in Section 4.4, before we state and prove the main result of this chapter.

4.3.1 Global Sobolev estimates on the double-null coordinates

We begin by adapting the global Sobolev estimates of Chapter 2 to the gauge choice of the perturbation ϕ. More precisely, we record the estimates on hyperbolas in Minkowski space as described by the *double-null coordinate system* $(u, \underline{u}, \hat{x})$ with the metric (4.2.3).

Consider the set $\mathcal{I}^+ \stackrel{\text{def}}{=} \{u > 0, \underline{u} > 0, 2u\underline{u} - |\hat{x}|^2 > 0\}$. This set corresponds to the *interior of the future light cone* emanating from the origin in Minkowski space. On this set, we can define the time function

$$\tau \stackrel{\text{def}}{=} \sqrt{2u\underline{u} - |\hat{x}|^2}. \tag{4.3.1}$$

Notation 4.3.1. The level set of τ will be denoted by Σ_τ. The Riemannian metric induced on Σ_τ by the Minkowski metric $\mathfrak{m}$ will be denoted η_τ. The geometric metric g also induces a symmetric bilinear form on Σ_τ, we will denote it by h_τ. When h_τ can *in principle* be Lorentzian or degenerate, in our application it will turn out to be always Riemannian.

Remark 4.3.2. It is straightforward to check that (Σ_τ, h_τ) as described above is isometric to the hyperboloid defined in (2.1.1) equipped with the metric (2.1.4).

We introduce also the hyperbolic radial function ρ within this forward light-cone $\mathcal{I}^+$ by

$$\rho \stackrel{\text{def}}{=} \cosh^{-1}\left(\frac{u+\underline{u}}{\sqrt{2}\tau}\right). \tag{4.3.2}$$

We note that relative to the Minkowski metric, the unit normal to Σ_τ is given by (using an abuse of notation)

$$-(\mathrm{d}\tau)^\sharp = \frac{1}{\tau}\left(u\partial_u + \underline{u}\partial_{\underline{u}} + \hat{x}\cdot\partial_{\hat{x}}\right). \tag{4.3.3}$$

Relative to the perturbed metric g, the unit normal to Σ_τ takes the form

$$-\frac{(\mathrm{d}\tau)^{g\sharp}}{\sqrt{|g(\mathrm{d}\tau,\mathrm{d}\tau)|}} = -\frac{(\mathrm{d}\tau)^\sharp + 2(\frac{u}{\tau})\phi\Upsilon''\partial_{\underline{u}} - \mathring{g}(\mathrm{d}\phi,\mathrm{d}\tau)\partial^{g\sharp}\phi}{\sqrt{1-2(\frac{u}{\tau})^2\phi\Upsilon'' + |g|^{-1}[\mathring{g}(\mathrm{d}\phi,\mathrm{d}\tau)]^2}}. \tag{4.3.4}$$

We define the following vector fields:

$$T = \frac{1}{\sqrt{2}}(\partial_u + \partial_{\underline{u}}); \tag{4.3.5}$$

$$L^1 = u\partial_u - \underline{u}\partial_{\underline{u}}; \tag{4.3.6}$$

$$L^i = \frac{1}{\sqrt{2}}(u+\underline{u})\partial_{\hat{x}^i} + \frac{1}{\sqrt{2}}\hat{x}^i(\partial_u + \partial_{\underline{u}}), \quad i = 2,\ldots,d. \tag{4.3.7}$$

It is straightforward to check that T is the future timelike vector field ∂_t of Minkowski space *when expressed in the double-null coordinates*. Similarly, L^i in (4.3.6) –(4.3.7) are the Lorentzian boosts $x^i\partial_t + t\partial_{x^i}$ when expressed in $(u,\underline{u},\hat{x})$. Hence, these vector fields are all Killing with respect to the Minkowski metric. Note that the L^i (where $i = 1,\ldots,d$) are also all tangential to Σ_τ. If α is an m-tuple with elements drawn from $\{1,\ldots,d\}$ (namely that $\alpha = (\alpha_1,\ldots,\alpha_m)$ with $\alpha_i \in \{1,\ldots,d\}$), we denote by L^α the differential operator

$$f \mapsto L^{\alpha_m}L^{\alpha_{m-1}}\cdots L^{\alpha_2}L^{\alpha_1}f.$$

By $|\alpha|$ we refer to its length, namely m.

The global Sobolev inequality adapted to the double-null coordinates on Σ_τ reads:

Theorem 4.3.3 (Global Sobolev inequality in double-null coordinates)**.** *Let $\ell \in \mathbb{R}$ be fixed. For any function f defined on $\mathcal{I}^+$, we have, for any $(u, \underline{u}, \hat{x}) \in \mathcal{I}^+$,*

$$|f(u, \underline{u}, \hat{x})|^2 \lesssim_{d,\ell} \tau_0^{-d} \cosh(\rho_0)^{1-d-\ell} \sum_{|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1} \int_{\Sigma_{\tau_0}} \cosh(\rho)^\ell |L^\alpha f|^2 \, \mathrm{dvol}_{\eta_{\tau_0}}.$$

The quantities τ_0 and ρ_0 appearing on the right of the inequality are given as

$$\tau_0 = \tau(u, \underline{u}, \hat{x}), \qquad \rho_0 = \rho(u, \underline{u}, \hat{x}).$$

Proof. The estimate follows immediately from Theorem 2.1.1 when noting that the weight function w_τ (see (2.1.3)) in the weighted Sobolev space $\mathcal{W}_\ell^{\lfloor d/2 \rfloor + 1, 2}$ is exactly $(u + \underline{u})/\sqrt{2}$ in the double-null coordinates. □

Remark 4.3.4. Note that by the definition of the function ρ, the coefficient in Theorem 4.3.3 can be written as

$$\tau_0^{-d} \cosh(\rho_0)^{1-d-\ell} = \tau_0^{\ell-1} \left(\frac{u + \underline{u}}{\sqrt{2}} \right)^{1-d-\ell}.$$

4.3.2 A weighted vector field algebra

In classical arguments using the vector field method, one typically commutes the equation with the generators of the Poincaré group, which consists of the

- translation vector fields $\partial_t, \partial_{x^i}$;
- rotations $x^i \partial_{x^j} - x^j \partial_{x^i}$;
- Lorentz boosts $t\partial_{x^i} + x^i \partial_t$.

These vector fields form, under the Lie bracket, an $\mathbb{R}$-algebra.

In applying the Sobolev inequality of the previous section, we intend to only commute with the Lorentz boosts L^i. This subset does not form an $\mathbb{R}$-algebra under the Lie bracket.

However, they form an algebra with coefficients drawn from a space of weights. For convenience, we introduce the y-coordinates

$$y^0 = \frac{u + \underline{u}}{\sqrt{2}}, \quad y^1 = \frac{u - \underline{u}}{\sqrt{2}}, \quad y^i = \hat{x}^i \ (i \geq 2). \tag{4.3.8}$$

Definition 4.3.5. We denote by $\mathfrak{W}_*$ the (commutative) ring of polynomial expressions in the $d+1$ variables $\left\{\frac{y^1}{y^0}, \ldots, \frac{y^d}{y^0}, \frac{1}{y^0}\right\}$, with $\mathbb{R}$ coefficients. This ring can be graded according to the degree of the $\frac{1}{y^0}$ term in the polynomial expression, we denote by $\mathfrak{W}_i$ the corresponding set of homogeneous elements.

Remark 4.3.6. By way of clarification and for example, we will have that the expression $\frac{y^1}{y^0} \in \mathfrak{W}_0$, while $(\frac{1}{y^0})^5(\frac{y^2}{y^0})(\frac{y^4}{y^0}) \in \mathfrak{W}_5$.

Remark 4.3.7. Notice that within the light cone $\mathcal{I}^+$, we have that the functions (for all $i = 1, \ldots, d$)

$$\left|\frac{y^i}{y^0}\right| \leq 1$$

are uniformly bounded.

Now, observe that for $i, j \in \{1, \ldots, d\}$,

$$T\left(\frac{1}{y^0}\right) = -\left(\frac{1}{y^0}\right)^2, \qquad T\left(\frac{y^i}{y^0}\right) = -\frac{1}{y^0} \cdot \frac{y^i}{y^0},$$

$$L^i\left(\frac{1}{y^0}\right) = -\frac{1}{y^0} \cdot \frac{y^i}{y^0}, \qquad L^i\left(\frac{y^j}{y^0}\right) = \delta_{ij} - \frac{y^i}{y^0} \cdot \frac{y^j}{y^0}.$$

Furthermore,

$$[L^i, T] = -\frac{1}{y^0} L^i + \frac{y^i}{y^0} T, \qquad [L^i, L^j] = \frac{y^i}{y^0} L^j - \frac{y^j}{y^0} L^i.$$

Together these implies that the set of vector fields of the form $c_0 T + \sum c_i L^i$ where the c_μ are taken from $\mathfrak{W}_*$ form not only an $\mathbb{R}$-Lie algebra, but also an algebra over the ring $\mathfrak{W}_*$, with multiplication being the Lie bracket. We will denote this algebra by $\mathfrak{A}_*$. The following proposition follows immediately from the computations above.

Proposition 4.3.8. *For $i \in \mathbb{Z}_+$, define*

$$\mathcal{A}_0 \stackrel{def}{=} \left\{ \sum_{j=1}^{d} c_j L^j \mid c_j \in \mathfrak{W}_0 \right\}, \qquad \mathcal{A}_i = \left\{ c_0 T + \sum_{j=1}^{d} c_j L^j \mid c_0 \in \mathfrak{W}_{i-1},\ c_j \in \mathfrak{W}_i \right\}.$$

Then $\mathcal{A}_*$ is graded, with $L^i \in \mathcal{A}_0$, and $T \in \mathcal{A}_1$. In particular, given elements $X_a \in \mathcal{A}_a, X_b \in \mathcal{A}_b$ and $f \in \mathfrak{W}_c$, we have that

$$[X_a, X_b] \in \mathcal{A}_{a+b}, \quad f X_a \in \mathcal{A}_{a+c}.$$

Remark 4.3.9. *We remark that we also have the following commutator relation*

$$[L^i, [L^j, T]] = \delta_{ij} T \in \mathcal{A}_1$$

as expected.

Using $\mathcal{A}_*$, we can build an algebra of differential operators which we label by $\mathfrak{B}_*^{*,*}$. Consider terms of the form

$$f X^1 X^2 X^3 \dots X^k \tag{4.3.9}$$

where $f \in \mathfrak{W}_*$ and $X^\alpha \in \{L^i, T\}$. They are differential operators that act on functions defined on $\mathcal{I}^+$ in the usual way. Using the computations above we see that terms of such form are closed under composition of differential operators. Hence we define $\mathfrak{B}_*^{*,*}$ as the set of finite sums of terms of the form (4.3.9), with addition defined normally and composition as multiplication; $\mathfrak{B}_*^{*,*}$ is obviously a $\mathfrak{W}_*$-module.

In exactly the same way as $\mathcal{A}_*$, the algebra $\mathfrak{B}_*^{*,*}$ is graded. We will use its lower index to record this grading.

Definition 4.3.10. The *weight* of a term of the form (4.3.9), where f is a monomial, is defined by the number of times T appears among the X^α, plus the number of times $1/y^0$ appears in the monomial f. The *degree* of a term of the form (4.3.9) is defined as the number k. The T*-degree* of a term of the form (4.3.9) is the number of times T appears among the X^α. By $\mathfrak{B}_w^{k,s}$ we refer to the set of finite sums in $\mathfrak{B}_*^{*,*}$ of elements with weight w and degree *at most* k, and T-degree *at most* s.

Remark 4.3.11. The set $\mathfrak{B}^{k,s}_w$ is well-defined due to Proposition 4.3.8. One needs to check that, for example, fX^1X^2 and $fX^2X^1 + f[X^1,X^2]$, which are equal as differential operators, have the same degrees and weight. Proposition 4.3.8 implies that for $X^\alpha \in \{L^i, T\}$, the terms making up $[X^1,X^2]$ always have the same weight as X^1X^2, and with same or lower T-degree.

For example, given any m-tuple α, the operator $L^\alpha \in \mathfrak{B}^{m,0}_0$, while we can identify $\mathfrak{W}_* = \mathfrak{B}^{0,0}_*$. The set $\mathfrak{A}_w$ are the set of degree (exactly) 1 elements in $\mathfrak{B}^{*,*}_w$. The following proposition follows immediately from the definition and Proposition 4.3.8.

Proposition 4.3.12. *If $A \in \mathfrak{B}^{k,s}_w$, and $B \in \mathfrak{B}^{k',s'}_{w'}$, then*

1. $AB \in \mathfrak{B}^{k+k',s+s'}_{w+w'}$;
2. $[A,B] \in \mathfrak{B}^{k+k'-1,s+s'}_{w+w'}$.

We remark finally that if $f = f(u)$ is a function defined within the light cone $\mathcal{I}^+$, then

$$Tf = \frac{1}{\sqrt{2}} f'(u), \quad \text{and } L^1 f = u f'(u).$$

In particular, if f is smooth and supported within a slab $u \in (a,b)$, then both Tf and $L^1 f$ are functions of u alone that are smooth and supported within $u \in (a,b)$. Similarly, we see that for $i \geq 2$

$$L^i f = \frac{1}{\sqrt{2}} \hat{x}^i f'(u).$$

To estimate functions of this form, we will use the following lemma.

Lemma 4.3.13. *Fix $f = f(u)$ a smooth function supported in $u \in [a,b]$. Then on the set $\mathcal{I}^+$, for any m-tuple α, we have*

$$|L^\alpha f| \lesssim (1 + \underline{u})^{m/2} \cdot \mathbf{1}_{\{u \in [a,b]\}}.$$

The implicit constant depends on the numbers a, b, the degree m, the dimension d, as well as $\|f\|_{C^m}$.

Proof. Observe that if $i, j \in \{2, \ldots, d\}$,

$$L^i \hat{x}^j = \frac{1}{\sqrt{2}} \delta^{ij} (u + \underline{u})$$

and that

$$L^i (u + \underline{u}) = \sqrt{2} \hat{x}^i.$$

So we have that up to a universal structural constant depending only on the dimension d and the degree m,

$$|L^\alpha f| \lesssim \left(1 + |u|^m + |u + \underline{u}|^{m/2} + |\hat{x}|^m\right) \cdot \|f\|_{C^m} \cdot \mathbf{1}_{\{u \in [a,b]\}}.$$

As on the set of interest, $u \in [\max(a,0), b]$, we have that $|u| < b$. Furthermore, on $\mathcal{I}^+$ by definition we have $2u\underline{u} > |\hat{x}|^2$. The boundedness of u implies that $|\hat{x}| \lesssim \sqrt{\underline{u}}$. The desired bound follows. □

4.3.3 Generalized energy

In Subsection 2.2.1, for a multiplier vector field X and a solution to the linear wave equation $\Box_{\mathfrak{m}} \phi = 0$, we defined the X-energy of ϕ along spacelike hypersurfaces of Minkowski space (see (2.2.7)). This was done through the energy-momentum tensor $Q[\phi]$, a symmetric $\binom{0}{2}$-tensor adapted to the Minkowski metric $\mathfrak{m}$. Motivated by the quasilinear nature of the perturbed metric g (see (4.2.9)), we now generalize this notion of energy to arbitrary metrics.

In the flowing discussion, let $\mathbf{g}$ denote any Lorentzian metric. Then we define the *energy-momentum tensor associated to* ϕ *and with respect to* $\mathbf{g}$,

$$Q[\phi; \mathbf{g}] \stackrel{\text{def}}{=} \mathrm{d}\phi \otimes \mathrm{d}\phi - \frac{1}{2} \mathbf{g}(\mathrm{d}\phi, \mathrm{d}\phi) \mathbf{g}. \tag{4.3.10}$$

Given a multiplier vector field X, we define the corresponding *X-energy current with respect to* $\mathbf{g}$ to be the vector field

$${}^{(X)}\mathcal{J}^\alpha[\phi; \mathbf{g}] \stackrel{\text{def}}{=} (\mathbf{g}^{-1})^{\alpha\beta} Q_{\beta\gamma}[\phi] X^\gamma. \tag{4.3.11}$$

The following identity is the analogue of the divergence identity (2.2.8) with respect to $\mathbf{g}$, which follows from standard computations:

$$\mathrm{div}_{\mathbf{g}}\left({}^{(X)}\mathcal{J}[\phi;\mathbf{g}]\right) = \Box_{\mathbf{g}}\phi X\phi + \frac{1}{2}Q^{\alpha\beta}[\phi;\mathbf{g}]\mathcal{L}_X\mathbf{g}_{\alpha\beta}. \tag{4.3.12}$$

Specializing now to the cases where $\mathbf{g}$ is either the Minkowski metric $\mathfrak{m}$ or the perturbed metric g, we define the following T-energies along Σ_τ:

$$\mathcal{E}_\tau[\phi]^2 \stackrel{\text{def}}{=} \mathcal{E}_\tau[\phi;\mathfrak{m}]^2 \stackrel{\text{def}}{=} 2\int\limits_{\Sigma_\tau} \mathcal{Q}[\phi;\mathfrak{m}](T,-(\mathrm{d}\tau)^\sharp)\ \mathrm{dvol}_{\eta_\tau}, \tag{4.3.13}$$

$$\mathcal{E}_\tau[\phi;g]^2 \stackrel{\text{def}}{=} 2\int\limits_{\Sigma_\tau} \frac{1}{\sqrt{|g(\mathrm{d}\tau,\mathrm{d}\tau)|}}\mathcal{Q}[\phi;g](T,(-\mathrm{d}\tau)^{g\sharp})\ \mathrm{dvol}_{h_\tau}, \tag{4.3.14}$$

We emphasize that the one corresponding to the Minkowski metric can be written explicitly as

$$\mathcal{E}_\tau[\phi]^2 = \int\limits_{\Sigma_\tau} \frac{1}{\tau^2\cosh\rho}\sum_{i=1}^{d}(L^i\phi)^2 + \frac{1}{\cosh\rho}(T\phi)^2\ \mathrm{dvol}_{\eta_\tau}, \tag{4.3.15}$$

see the discussion immediately after the definitions of T and L^i (4.3.5) – (4.3.7), Lemma 2.2.5, and its proof via the identity (A.2.17).

Integrating the divergence identity (4.3.12) with $\mathfrak{m}$ and g in place of $\mathbf{g}$ and T in place of X between two level sets $\tau_0 < \tau_1$ of τ one obtains[4] the *energy inequalities*

$$\mathcal{E}_{\tau_1}[\phi;g]^2 - \mathcal{E}_{\tau_0}[\phi;g]^2 \lesssim \iint\limits_{\tau\in[\tau_0,\tau_1]} \left|\Box_g\phi\cdot T(\phi)\right| + \left|\mathcal{Q}[\phi;g] :_g \mathcal{L}_T g\right|\ \mathrm{dvol}_g, \tag{4.3.16}$$

$$\mathcal{E}_{\tau_1}[\phi]^2 - \mathcal{E}_{\tau_0}[\phi]^2 \lesssim \iint\limits_{\tau\in[\tau_0,\tau_1]} |\Box\phi\cdot T\phi|\ \mathrm{dvol}_{\mathfrak{m}}. \tag{4.3.17}$$

The following proposition is how one obtains pointwise control for terms appearing on the right hand side of (4.3.16) and (4.3.17).

[4]Note that the future oriented normals Σ_τ with respect to $\mathfrak{m}$ and g are (4.3.3) and (4.3.4), respectively. These appear when applying the divergence theorem to the left hand side of (4.3.12) through the definitions (4.3.11), (4.3.13), and (4.3.14).

Proposition 4.3.14. *For any function f defined on $\mathcal{I}^+$, we have, for any $(u, \underline{u}, \hat{x}) \in \mathcal{I}^+$,*

$$|f(u,\underline{u},\hat{x})| \lesssim_d \tau^{1-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}} \sum_{|\alpha| \leq \lfloor \frac{d}{2} \rfloor} \mathcal{E}_\tau[L^\alpha f],$$

$$|L^i f(u,\underline{u},\hat{x})| \lesssim_d \tau^{1-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}} \sum_{|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1} \mathcal{E}_\tau[L^\alpha f],$$

$$|T f(u,\underline{u},\hat{x})| \lesssim_d \tau^{-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}} \sum_{|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1} \mathcal{E}_\tau[L^\alpha f].$$

Proof. This follows immediately when Propositions 2.2.7 (with $M = 0$) and 2.2.9 are expressed in double-null coodinates through Theorem 4.3.3. □

Remark 4.3.15 (Remark 2.2.6 expressed in double-null coordinates.)**.** A feature of the energy (4.3.15) is its *anisotropy*. The classical energy estimates of wave equations control integrals of $|\partial_t \phi|^2 + |\nabla \phi|^2$ where all components appear on equal footing. Here, however, the transversal (to Σ_τ) derivative $T\phi$ has a different weight compared to the tangential derivatives $L^i\phi$. Noting that by their definitions, T has *unit-sized* coefficients with expressed relative to the standard coordinates of Minkowski space. The coefficients for L^i (within the light cone $\mathcal{I}^+$) have size $\approx t$. Therefore an *isotropic* analogue would be expected to contain integrals of $\frac{1}{t^2}(L^i\phi)^2$ along with integrals of $T\phi$. Noting that $t = \tau \cosh \rho$ this indicates that an isotropic analogue would contain, instead of the integral given in (4.3.15), the integral

$$\int_{\Sigma_\tau} \frac{1}{\tau^2 \cosh(\rho)^3} \sum (L^i \phi)^2 + \frac{1}{\cosh(\rho)} (T\phi)^2 \, \mathrm{dvol}_{\eta_\tau}.$$

In other words, the integral for $L^i\phi$ in the energy has a *better* ρ weight than would be expected from an isotropic energy, such as that controlled by the standard energy estimates.

This improvement reflects the fact that the energy estimate described in this section captures the *peeling properties* of linear waves within the energy integral itself. It is well-known that derivatives *tangential to an out-going light-cone* decay faster along the light-cone, than derivatives transverse to the light-cone. As asymptotically hyperboloids approximate light-cones, we expect the same peeling property to survive. Indeed, the energy inequality (4.3.16) shows that we can capture this in the integral sense.

4.4 A semilinear model

Before stating and proving our main results, we will illustrate both our method of proof and the main difficulties encountered in the simpler setting of a *semilinear* problem. Recall that the small-data global existence problem for the membrane equation (4.1.1) in dimension $d \geq 3$ follows from a direct application of Klainerman's vector field method, after noting that the equation of motion is a quasilinear perturbation of the linear wave equation with *no* quadratic nonlinearities, see the expository book [Sog08]. In particular, Klainerman's null condition plays no role in establishing this result. As indicated in Remark 4.2.3, the perturbation problem for simple planewaves introduces resonant quadratic terms to which Klainerman's null condition does not directly apply. On the other hand, as observed in that same remark, there is a hidden null structure from which we can expect to recover some improved decay rates.

The main difficulty however is that Klainerman's null condition is built upon the *expected decay rates* corresponding to solutions to the linear wave equations with *strongly localized initial data*. In particular, the heuristic for the null condition is based on the expectation that, for generic first derivatives of such a solution, $\partial\phi$ decays like $t^{(1-d)/2}$; while for "tangential" (to an outgoing null cone foliation) derivatives, the corresponding derivatives decays like $t^{-d/2}$.

In our setting, however, one of the waves in the interaction is a simple planewave which *does not decay at all*. This reduces the effectiveness of the null structure in improving decay. As will be shown this difficulty manifests already in the model problem to be discussed in this section. For example, when applying the vector field method one studies the equations of motion satisfied by higher derivatives of the solution. After commuting the equation with the Lorentz boosts, one sees that when the boost hits on Υ'', we obtain a coefficient that, while still localized to $t \approx -x^1$, is growing in time. On an intuitive level one can interpret this as a transfer of energy from the (infinite energy) background simple planewave to the perturbation. The null structure in our context then serves to cap the rate of this energy transfer, ensuring (in our case) global existence of the perturbed solution.

The specific semilinear model problem we consider takes (4.2.17) and drops from it the quasilinearity. That is to say, we consider the small-data problem for the semilinear wave equation

$$\Box\phi = \Upsilon''(u)(\phi_{\underline{u}})^2 \tag{4.4.1}$$

on $\mathbb{R}^{1,d}$, where $\Box$ is the usual wave operator corresponding to the Minkowski metric $\mathfrak{m}$. To approach this problem using a vector field method, one commutes (4.4.1) with the Lorentz boosts to derive equations of motions for higher order derivatives. The energy estimates for these higher order derivatives are then combined with the global Sobolev inequality to get L^∞ *decay* estimates for the solution. The main difficulty one encounters here, however, is when the vector fields hit on the background Υ''. We have

$$\Box L^\alpha\phi = L^\alpha(\Upsilon'')(\phi_{\underline{u}})^2 + \dots,$$

where $L^\alpha(\Upsilon'')$ can have *growing* L^∞ norm.

This potential growth of the coefficients is the main technical complication in this problem. The best uniform estimate we have for $L^\alpha(\Upsilon'')$, assuming for convenience $\Upsilon \in C_0^\infty$ and the initial data for ϕ is compactly supported, is via Lemma 4.3.13, which gives

$$\Box L^\alpha\phi \approx (1+\underline{u})^{m/2}\mathbf{1}_{\{u\in[a,b]\}}\cdot\underbrace{(1+u+\underline{u})^{-d/2}}_{\partial_{\underline{u}}\phi}\partial\phi$$

where we have made the optimistic assumption that $\phi_{\underline{u}}$ decays like $(1+u+\underline{u})^{-d/2}$, as would be the case for a linear wave.

At this point, **two different complications present themselves**. First, one may naively hope that the (higher order) energies always stay bounded, in analogy with the linear case. This hope is rapidly dashed when we examine the energy estimate for $|\alpha| = d$. After commuting with d derivatives, we see that

$$\Box L^\alpha \phi \approx \partial\phi$$

with no decay! Even assuming that we can prove the boundedness of the lowest order energy (which controls $\partial\phi$ in L^2), the best we can obtain is then that energy for $L^\alpha\phi$ grows linearly in time. This first difficulty can be **overcome with a modified bootstrap scheme** where the expected (polynomial in time) energy growth is incorporated into the assumptions.

Remark 4.4.1. Several remarks are in order concerning this energy growth:

1. This growth is different from what appears in typical applications of the vector field method to nonlinear wave equations with null-condition satisfying nonlinearities in $d = 3$. In those cases, the equation takes the schematic form

   $$\Box\phi = L\phi\underline{L}\phi$$

   where $L\phi$ is a "good" derivative that is expected to decay like $t^{-d/2}$ and $\underline{L}\phi$ is a "bad" derivative that is expected to decay like $t^{(1-d)/2}$. When considering the energy estimates for the top order derivatives, one must face the possibility of needing to control

   $$\Box\partial^\alpha\phi = \partial^\alpha L\phi \cdot \underline{L}\phi + \ldots.$$

   To close the energy estimate, one must estimate $\partial^\alpha L\phi$ in L^2 and thereby bound $\underline{L}\phi$ in L^∞ by $1/t$, whereupon the time integration gives a small energy growth of the top order derivatives.

   This difficulty is already largely avoided in hyperboloidal energy methods, exploiting the anisotropic inclusion of "good" versus "bad" derivatives in the energy (see

Proposition **??** and Remark 4.3.15), and is *not* the cause of the energy growth in our argument.

2. That the two energy growths are distinct can be seen in the fact that for the classical applications of the vector field method, the energy growth occurs only for the highest order derivatives used in the argument. The more derivatives one uses in the bootstrap, the more levels of energy that remain bounded. In our case, the energy growth starts appearing at a fixed (depending on the dimension d) level of derivatives, regardless of how many derivatives is used in the bootstrap argument. The reason for this is because there are terms in the equation that do not enjoy the boost symmetries, and every time you differentiate them with a boost, one gets another growth of $\underline{u}$. (see Lemma 4.3.13).

3. Similarly, this energy growth is also different from the μ-degeneracy of the highest orders of energies (and the associated "descent scheme") that appears in the study of formation of shocks [Chr07] (see also discussion in [HKSW16]).

The second difficulty is more sinister. To close the energy estimate, and estimate $\phi_{\underline{u}}$ in L^∞, we need to commute with at least $d/2$ derivatives in order to make use of Sobolev, implying that $m > d/2$. But then the coefficients on the right hand side are of size $(1+\underline{u})^{-d/4+\epsilon}$, which is not integrable when $d = 2,3,4$. This seemingly prevents us from even closing *any* bound for $|\partial\phi|$. Take for example the case $d = 3$.

- Assuming L^∞ control on $|\partial\phi|$ of the type $(1+u+\underline{u})^\lambda$, the coefficients in the equation for $LL\phi$ grows like $(1+\underline{u})^{1-\lambda}$, This implies that, even assuming the lowest-order energy remains bounded, the energy for $LL\phi$ grows like $(1+u+\underline{u})^{2-\lambda}$.

- In the *best* case, we expect that the growth of the $LL\phi$ energy means a weakened L^∞ control on $|\partial\phi|$, to the tune of $(1+u+\underline{u})^{2-\lambda-3/2}$, with the power $(-3/2)$ coming from the global Sobolev inequalities.

- Thus, we see that *at every iteration* one would increase the growth rate of $|\partial\phi|$ by $(1 + u + \underline{u})^{1/2}$.

To handle this difficulty, **we will make use of the hyperboloidal foliation and its associated sharp global Sobolev inequalities**. In particular, the anisotropy discussed in Remark 4.3.15 allows us to exploit an additional vestige of the null structure of the membrane equation to gain, effectively, an additional $(1 + u + \underline{u})^{-1}$ decay in the most difficult terms and close the argument also in $d = 3$ and 4. This is accomplished by essentially "borrowing" a weight from the $|\partial_{\underline{u}}\phi|$ term when we put it in L^2, using the fact that the term we are trying to control is also a "good derivative" and benefits from the anisotropic energy. The vestigial null structure is explained in Remark 4.5.3 below.

Remark 4.4.2. This improvement is not sufficient for the $d = 2$ case, even at the heuristic level, due to logarithmic divergences when integrating $(1 + s)^{-1}$. As the stability of planewaves is trivial in $d = 1$ (using either the integrability of the membrane equation in this case, or via an easy modification of the arguments in [Won17b]), we have reasons to expect that the stability result also holds for $d = 2$. This turns out to be indeed the case, if we factor in the additional improvements we used in the more detailed analyses for the quasilinear problem in Section 4.5. See also Remark 4.5.1.

Note that these difficulties are essentially due to the fact that the background function $\Upsilon(u)$, while being a solution to the linear wave equation $\Box\Upsilon(u) = 0$, is not one that is associated to localized initial data. Hence its derivative with respect to L^α has worse decay rates. (In fact, it grows in time.)

Concerning this semilinear model, we will study the initial value problem for (4.4.1) with initial data prescribed on the hypersurface $\{y^0 = 2\}$ (here y is defined as in (4.3.8)),

$$\phi|_{y^0=2} = \phi_0, \quad \partial_{y^0}\phi|_{y^0=2} = \phi_1.$$

The remainder of this section is devoted to proving the following theorem.

Theorem 4.4.3. *Let $d \geq 3$ and assume $\Upsilon''(u)$ is smooth and has compact support in u. Consider the initial value problem for (4.4.1) where ϕ_0 and ϕ_1 are smooth compactly supported functions on $B(0,1) \subset \mathbb{R}^d$. Let $s = d$ if d is odd, and $s = d+1$ if d is even. Then provided $\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s}$ is sufficiently small, the initial value problem has a global-in-time solution.*

4.4.1 Preliminaries

Using the standard local existence theorem with finite-speed of propagation we can assume the solution exists up to at least Σ_2. Furthermore, by finite speed of propagation, the solution must vanish when

$$\sqrt{\sum_{i=1}^{d} |y^i|^2} > |y^0 - 2| + 1.$$

In particular, this implies

$$\sqrt{2}(u + \underline{u}) \leq \tau^2 + 1 \tag{4.4.2}$$

on the support of ϕ.

By the blow-up criterion for wave equations, it suffices to show $\|\phi\|_{W^{1,\infty}(\Sigma_\tau)} < \infty$ for every $\tau \in (2, \infty)$, see [Sog08, Chapter 1, Theorem 4.3]. The general approach, which we will take also for studying the quasilinear problem, is that of a bootstrap argument.

1. We will assume that, up to time $\tau_{\max} > 2$, that the energy $\mathcal{E}_\tau$ of the solution ϕ and its derivatives $L^\alpha \phi$ verify certain bounds.

2. Using Proposition 4.3.14, this gives L^∞ bounds on ϕ, and its derivatives of the form $L^\alpha \phi$ and $TL^\alpha \phi$.

3. We can then estimate the nonlinearity using these L^∞ estimates, which we then feed back into the energy inequality (4.3.16) to get an *updated* control on $\mathcal{E}_\tau$ for all $\tau \in [2, \tau_{\max}]$.

4. Finally, show for sufficiently small initial data sizes, the updated control *improves* the original control, whereupon by the method of continuity the original bounds on $\mathcal{E}_\tau$ must hold for all $\tau \geq 2$, implying the desired global existence.

Before implementing the bootstrap in the following two sections (one each for the cases d being odd or even), we record first basic pointwise bounds on the nonlinearity. For estimating the nonlinearity, we observe that

$$\partial_{\underline{u}} = \frac{\sqrt{2}u}{u+\underline{u}}T - \frac{1}{u+\underline{u}}L^1 = \frac{1}{u+\underline{u}}(\sqrt{2}uT - L^1). \tag{4.4.3}$$

This allows us to decompose

$$\Upsilon''(u)(\phi_{\underline{u}})^2 = \frac{1}{(u+\underline{u})^2}\Big[A(u)(L^1\phi)^2 + B(u)L^1\phi\cdot T\phi + C(u)(T\phi)^2\Big]$$

where A, B, C are all compactly supported smooth functions of u. By Proposition 4.3.8 we can rewrite $L^\alpha T\phi$ as

$$L^\alpha T\phi = \sum_{|\beta|\leq|\alpha|}\frac{1}{u+\underline{u}}c_\beta L^\beta\phi + \sum_{|\gamma|\leq|\alpha|}c'_\gamma TL^\gamma\phi$$

where $c_\beta, c'_\gamma \in \mathfrak{W}_0$ and hence are bounded. Additionally on the region $\tau \geq 2$ that we are interested in, $u+\underline{u}$ is bounded from below. So finally using Lemma 4.3.13 on the coefficients A, B, C above, we obtain the following uniform pointwise bound on the region $\{\tau \geq 2\}$

$$\begin{aligned}\Big|L^\alpha\Big[\Upsilon''(u)(\phi_{\underline{u}})^2\Big]\Big| \lesssim \sum_{k+\ell_1+\ell_2\leq|\alpha|}(1+\underline{u})^{\frac{k}{2}-2}\cdot\mathbf{1}_{\{u\in\operatorname{supp}\Upsilon''\}}\cdot \\ \Big|(L^{\leq\ell_1+1}\phi)(L^{\leq\ell_2+1}\phi) + (TL^{\leq\ell_1}\phi)(TL^{\leq\ell_2}\phi) + (L^{\leq\ell_1+1}\phi)(TL^{\leq\ell_2}\phi)\Big|.\end{aligned} \tag{4.4.4}$$

Notation 4.4.4. *Here we denote schematically by $L^{\leq\ell}\phi$ terms of the form $L^\beta\phi$ with β an m-tuple with $m \leq \ell$.*

By Proposition 4.3.14 we can replace the term with the smallest of ℓ_1, ℓ_2 using an energy integral:

$$\begin{aligned}
\left|(L^{\leq \ell_1+1}\phi)(L^{\leq \ell_2+1}\phi)\right| &\lesssim \tau^{1-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}} \mathcal{E}_\tau[L^{\leq \ell_1+\lfloor \frac{d}{2} \rfloor+1}\phi] \cdot \left|L^{\leq \ell_2+1}\phi\right|, \\
\left|(L^{\leq \ell_1+1}\phi)(TL^{\leq \ell_2}\phi)\right| &\lesssim \tau^{1-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}} \mathcal{E}_\tau[L^{\leq \ell_1+\lfloor \frac{d}{2} \rfloor+1}\phi] \cdot \left|TL^{\leq \ell_2}\phi\right|, \\
\left|(TL^{\leq \ell_1}\phi)(L^{\leq \ell_2+1}\phi)\right| &\lesssim \tau^{-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}} \mathcal{E}_\tau[L^{\leq \ell_1+\lfloor \frac{d}{2} \rfloor+1}\phi] \cdot \left|L^{\leq \ell_2+1}\phi\right|, \\
\left|(TL^{\leq \ell_1}\phi)(TL^{\leq \ell_2}\phi)\right| &\lesssim \tau^{-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}} \mathcal{E}_\tau[L^{\leq \ell_1+\lfloor \frac{d}{2} \rfloor+1}\phi] \cdot \left|TL^{\leq \ell_2}\phi\right|.
\end{aligned}$$

This allows us to condense (4.4.4) as

$$\begin{aligned}
\left|L^\alpha\left[\Upsilon''(u)(\phi_{\underline{u}})^2\right]\right| \lesssim \sum_{\substack{k+\ell_1+\ell_2 \leq |\alpha| \\ \ell_1 \leq \ell_2}} &(1+\underline{u})^{\frac{k}{2}-2} \cdot \mathbf{1}_{\{u \in \operatorname{supp} \Upsilon''\}} \cdot \\
&\underbrace{(1+\underline{u})}_{\approx \tau \cosh(\rho)}{}^{2-\frac{d}{2}} \cdot \mathcal{E}_\tau[L^{\leq \ell_1+\lfloor \frac{d}{2} \rfloor+1}\phi]\left[\frac{1}{\tau \cosh(\rho)}|L^{\leq \ell_2+1}\phi| + \frac{1}{\cosh(\rho)}|TL^{\leq \ell_2}\phi|\right]. \quad (4.4.5)
\end{aligned}$$

Here we used that $\tau \cosh(\rho) = \frac{1}{\sqrt{2}}(u+\underline{u}) \approx (1+\underline{u})$ using the support properties of Υ''. Next we note that $2u\underline{u} \geq \tau^2$ in $\mathcal{I}^+$. On the support of Υ'' this means $\underline{u} \gtrsim \tau^2$. On the other hand, from (4.4.2) we also get $\underline{u} \lesssim 1+\tau^2$. This allows us to replace $(1+\underline{u})$ by $(1+\tau)^2$ in (4.4.5).

Observe next that since L^i is Killing with respect to $\mathfrak{m}$, we have that firstly $\mathcal{L}_T \mathfrak{m} = 0$ and secondly

$$\left|\Box L^\alpha \phi\right| \leq \left|L^\alpha\left[\Upsilon''(u)(\phi_{\underline{u}})^2\right]\right|.$$

So from the energy identity (4.3.16) we get

$$\mathcal{E}_{\tau_1}[L^\alpha \phi]^2 - \mathcal{E}_{\tau_0}[L^\alpha \phi]^2 \lesssim \iint\limits_{\tau \in [\tau_0, \tau_1]} |\Box L^\alpha \phi| \cdot |TL^\alpha \phi| \, \mathrm{dvol}_{\mathfrak{m}}.$$

Applying (4.4.5) we finally arrive at our fundamental *a priori* estimate

$$\begin{aligned}
&\mathcal{E}_{\tau_1}[L^\alpha \phi]^2 - \mathcal{E}_{\tau_0}[L^\alpha \phi]^2 \lesssim \\
&\qquad \int_{\tau_0}^{\tau_1} \sum_{\substack{k+\ell_1+\ell_2 \leq |\alpha| \\ \ell_1 \leq \ell_2}} (1+\tau)^{k-d} \cdot \mathcal{E}_\tau[L^\alpha \phi] \cdot \mathcal{E}_\tau[L^{\leq \ell_2}\phi] \cdot \mathcal{E}_\tau[L^{\leq \ell_1+\lfloor \frac{d}{2} \rfloor+1}\phi] \, \mathrm{d}\tau.
\end{aligned}$$

To simplify notation, let us write

$$\mathfrak{E}_k(\tau) = \sup_{\sigma\in[2,\tau]} \mathcal{E}_\sigma[L^{\leq k}\phi]. \tag{4.4.6}$$

Our *a priori* estimate reads

$$\mathfrak{E}_k(\tau)^2 - \mathfrak{E}_k(2)^2 \lesssim \sum_{\substack{\ell_0+\ell_1+\ell_2=k\\ \ell_1\leq\ell_2}} \int_2^\tau s^{\ell_0-d}\mathfrak{E}_k \mathfrak{E}_{\ell_1+\lfloor\frac{d}{2}\rfloor+1}\mathfrak{E}_{\ell_2}\mathrm{d}s. \tag{4.4.7}$$

In the remainder of this section we will discuss the bootstrap scheme that allows us to control $\mathfrak{E}_k$, for all $k \leq d+1$ when d is even and all $k \leq d$ when d is odd, for all time $\tau \geq 2$. Note that the implicit constant in (4.4.7) depends only on the dimension d, the order k of differentiation, and properties of the background function Υ, and is in particular independent of ϕ.

4.4.2 Bootstrap for $d \geq 6$ even

When $d \geq 6$ is even, we will denote by m the value $d/2$. Note that $m \geq 3$. We will assume a uniform bound on the initial data

$$\mathfrak{E}_k(2) \leq \epsilon, \quad k \leq d+1. \tag{4.4.8}$$

Our bootstrap assumption is that for some $\delta > \epsilon$ and for all $2 \leq \tau \leq \tilde{\tau}$,

$$\mathfrak{E}_k(\tau) \leq \begin{cases} \delta & k \leq d-2 \\ \delta\tau^{k-(d-1)}\ln(\tau) & d-1 \leq k \leq d+1 \end{cases}. \tag{4.4.9}$$

We note that under (4.4.7) this system is closed: if $\ell_0+\ell_1+\ell_2 \leq d+1$ and $\ell_1 \leq \ell_2$, then $\ell_1 \leq m$. This means that $\ell_1 + \lfloor d/2\rfloor + 1 \leq 2m+1 = d+1$. Our goal is to show that the bootstrap assumptions (4.4.9) can be used to prove improved versions of themselves, under a smallness assumption on δ and ϵ.

Under our bootstrap assumptions, we can expression every term of the form

$$s^{\ell_0-d}\mathfrak{E}_k(s)\mathfrak{E}_{\ell_1+m+1}(s)\mathfrak{E}_{\ell_2}(s) = w_{\ell_0,\ell_1,\ell_2}(s)\delta^3,$$

noting that $\ell_1 \le \ell_2$ by assumption and $\ell_0 + \ell_1 + \ell_2 = k \le d+1$. Observing that at most *one* of ℓ_0, $\ell_1 + m + 1$, and ℓ_2 can be $\ge d$ under these conditions, we tabulate upper bounds for the weight functions $w_{\ell_0,\ell_1,\ell_2}(s)$ in Table 4.1. From this table, we see immediately that

Table 4.1: ($d \ge 6$, even) List of admissible ℓ_0, ℓ_1, ℓ_2 values as well as the corresponding upper bounds for w_{ℓ_0,ℓ_1,ℓ_2}. The value of "—" means any value compatible with the prescribed columns. The shaded rows are those with non-integrable upper bounds for w_{ℓ_0,ℓ_1,ℓ_2}.

k	ℓ_0	ℓ_1	ℓ_2	$w_{\ell_0,\ell_1,\ell_2}(s) \le$	Comment
$\le d-2$	—	$< m-2$	—	s^{-2}	$\implies \ell_0 \le k$.
$\le d-2$	—	$m-2$	—	$s^{2-d}\ln(s)$	$\implies \ell_0 \le 2$.
$\le d-2$	—	$m-1$	—	$s^{-d}\ln(s)$	$\implies \ell_0 = 0$.
$d-1$	$\le d-2$	$\le m-3$	$\le d-2$	$s^{-2}\ln(s)$	
$d-1$	—	$m-2$	—	$s^{3-d}\ln(s)^2$	$\implies \ell_2 \le m+1, \ell_0 \le 3$
$d-1$	—	$m-1$	—	$s^{1-d}\ln(s)^2$	$\implies \ell_0 \le 1$
$d-1$	—	—	$d-1$	$s^{-d}\ln(s)^2$	
$d-1$	$d-1$	—	—	$s^{-1}\ln(s)$	
d	$\le d-2$	$\le m-3$	$\le d-2$	$s^{-1}\ln(s)$	
d	—	$m-2$	—	$s^{4-d}\ln(s)^3$	$\implies \ell_2 \le m+2, \ell_0 \le 4$
d	—	$m, m-1$	—	$s^{2-d}\ln(s)^2$	$\implies \ell_0 \le 2$
d	—	—	$d, d-1$	$s^{2-d}\ln(s)^2$	
d	$d, d-1$	—	—	$s\ln(s)$	
$d+1$	$\le d-2$	$\le m-3$	$\le d-2$	$\ln(s)$	
$d+1$	—	$m-2$	—	$s^{5-d}\ln(s)^3$	$\implies \ell_2 \le m+3, \ell_0 \le 5$
$d+1$	—	$m, m-1$	—	$s^{3-d}\ln(s)^3$	$\implies \ell_2 \le m+2, \ell_0 \le 3$
$d+1$	—	—	$d, d \pm 1$	$s^{4-d}\ln(s)^2$	
d	$d, d \pm 1$	—	—	$s^3\ln(s)$	

$\mathfrak{E}_k(\tau)^2 - \mathfrak{E}_k(2)^2 \lesssim \delta^3$ whenever $k \le d-2$. Furthermore, using that for $p > -1$

$$\int s^p \ln(s)\,\mathrm{d}s = \frac{1}{p+1}s^{p+1}\ln(s) - \frac{1}{(p+1)^2}s^{p+1} \lesssim s^{p+1}\ln(s)^2,$$

and

$$\int s^{-1}\ln(s)\,\mathrm{d}s = \frac{1}{2}\ln(s)^2,$$

we conclude that for $k = d-1, d, d+1$

$$\mathfrak{E}_k(\tau)^2 - \mathfrak{E}_k(2)^2 \lesssim \delta^3 \tau^{2(k-(d-1))}\ln(\tau)^2.$$

Thus for δ sufficiently small (depending on the implicit constants in the inequalities above), we have that our bootstrap assumptions (4.4.9) together with initial data assumptions implies

$$\mathfrak{E}_k(\tau) \leq \begin{cases} \sqrt{\epsilon^2 + \frac{1}{2}\delta^2} & k \leq d-2 \\ \sqrt{\epsilon^2 + \frac{1}{2}\delta^2 \tau^{2(k-(d-1))} \ln(\tau)^2} & d-1 \leq k \leq d+1 \end{cases}. \tag{4.4.10}$$

By choosing ϵ sufficiently small relative to δ, we can guarantee

$$\mathfrak{E}_k(\tau) \leq \begin{cases} \sqrt{\frac{3}{4}}\delta & k \leq d-2 \\ \sqrt{\frac{3}{4}}\delta \tau^{k-(d-1)} \ln(\tau) & d-1 \leq k \leq d+1 \end{cases}, \tag{4.4.11}$$

thereby closing the bootstrap and proving global existence.

4.4.3 Bootstrap for $d \geq 5$ odd

When $d \geq 5$ odd, we will take our bootstrap assumption to be

$$\mathfrak{E}_k(\tau) \leq \begin{cases} \delta & k \leq d-2 \\ \delta \tau^{k-(d-1)} \ln(\tau) & k = d-1, d \end{cases}. \tag{4.4.12}$$

That we can close with one fewer derivative is due to $\lfloor d/2 \rfloor < d/2$ in this case. Define $m = \lfloor d/2 \rfloor$ for convenience; note that $2m = d-1$. By our assumption then $\ell_1 \leq \ell_2 \implies \ell_1 \leq m$, and hence $\ell_1 + m + 1 \leq d$, allowing the system to close.

The bootstrap argument here is largely similar to the case $d \geq 6$ even. In Table 4.2 we record upper bounds for the weight functions $w_{\ell_0,\ell_1,\ell_2}(s)$, and omit the straightforward remainder of arguments.

4.4.4 Bootstrap for $d = 4$

We will assume a uniform bound on the initial data

$$\mathfrak{E}_k(2) \leq \epsilon, \quad k \leq 5, \tag{4.4.13}$$

Table 4.2: ($d \geq 5$, odd) List of admissible ℓ_0, ℓ_1, ℓ_2 values as well as the corresponding upper bounds for w_{ℓ_0,ℓ_1,ℓ_2}. The value of "—" means any value compatible with the prescribed columns. The shaded rows are those with non-integrable upper bounds for w_{ℓ_0,ℓ_1,ℓ_2}.

k	ℓ_0	ℓ_1	ℓ_2	$w_{\ell_0,\ell_1,\ell_2}(s) \leq$	Comment
$\leq d-2$	—	$\leq m-2$	—	s^{-2}	$\implies \ell_0 \leq k.$
$\leq d-2$	—	$m-1$	—	$s^{1-d}\ln(s)$	$\implies \ell_0 \leq 1.$
$d-1$	$\leq d-2$	$\leq m-2$	$\leq d-2$	$s^{-2}\ln(s)$	
$d-1$	—	$m-1$	—	$s^{2-d}\ln(s)^2$	$\implies \ell_0 \leq 2$
$d-1$	—	—	$d-1$	$s^{-d}\ln(s)^2$	
$d-1$	$d-1$	—	—	$s^{-1}\ln(s)$	
d	$\leq d-2$	$\leq m-2$	$\leq d-2$	$s^{-1}\ln(s)$	
d	—	$m, m-1$	—	$s^{4-d}\ln(s)^3$	$\implies \ell_0 \leq 3$
d	—	—	$d, d-1$	$s^{2-d}\ln(s)^2$	
d	$d, d-1$	—	—	$s\ln(s)$	

with an additional bootstrap assumption for some $\delta > \epsilon$

$$\mathfrak{E}_k(\tau) \leq \begin{cases} \delta & k \leq 2 \\ \delta\tau^{k-3+\gamma} & 3 \leq k \leq 5 \end{cases}. \tag{4.4.14}$$

The number γ is assumed to be $\ll 1$ and arbitrary; in particular we will throughout take $\gamma < \frac{1}{3}$. The smallness of γ will impact the smallness of the initial data allowed: the smaller the γ the smaller the initial data needs to be. We consider γ as fixed once and for all.

We argue similarly to the case when $d \geq 6$, and record in Table 4.3 the corresponding weight functions w_{ℓ_0,ℓ_1,ℓ_2}. Note, however, an additional complication arises since $d/2 + 1 = 3 = d - 1$ in this setting (which is why instead of a logarithmic growth of energy $\mathfrak{E}_{d-1}$, we see a small polynomial growth).

Based on the weights derived in the table, we see clearly that, by (4.4.7) we have

$$\mathfrak{E}_k(\tau)^2 - \mathfrak{E}_k(2)^2 \lesssim \begin{cases} \delta^3 & k \leq 2 \\ \delta^3\tau^{2\gamma+2k-6} & k = 3, 4, 5 \end{cases}$$

Table 4.3: ($d = 4$) List of admissible ℓ_0, ℓ_1, ℓ_2 values as well as the corresponding upper bounds for w_{ℓ_0,ℓ_1,ℓ_2}. The value of "—" means any value compatible with the prescribed columns. The shaded rows are those with non-integrable upper bounds for w_{ℓ_0,ℓ_1,ℓ_2}. (Recall that $3\gamma < 1$ by fiat.)

k	ℓ_0	ℓ_1	ℓ_2	$w_{\ell_0,\ell_1,\ell_2}(s) \leq$
≤ 2	—	0	—	$s^{\gamma-2}$
2	0	1	1	$s^{\gamma-3}$
3	≤ 2	0	≤ 2	$s^{2\gamma-2}$
3	—	1	—	$s^{2\gamma-2}$
3	0	0	3	$s^{3\gamma-4}$
3	3	0	0	$s^{2\gamma-1}$
4	4	0	0	$s^{2\gamma+1}$
4	—	—	1	$s^{2\gamma}$
4	—	—	2	$s^{2\gamma-1}$
4	—	—	3, 4	$s^{3\gamma-2}$
5	5	0	0	$s^{2\gamma+3}$
5	—	—	1	$s^{2\gamma+2}$
5	—	—	2	$s^{2\gamma+1}$
5	—	—	3,4,5	$s^{3\gamma}$

and hence taking δ sufficiently small and ϵ even sufficiently smaller will allow us to close the bootstrap and obtain global existence.

Remark 4.4.5. Applying Proposition 4.3.14 we see that the corresponding solution has the following decay rates:

$$|\phi| \lesssim (y^0)^{-1},$$

$$|L^i\phi| \lesssim (y^0)^{\gamma-1}, \qquad |L^iL^j\phi| \lesssim (y_0)^{\gamma},$$

$$|T\phi| \lesssim (y^0)^{\gamma-1}\tau^{-1}, \qquad |TL^i\phi| \lesssim (y^0)^{\gamma}\tau^{-1}.$$

The difference between the decay rate for $|\phi|$ and the expected $(y^0)^{-3/2}$ is due to our not using the Morawetz K multiplier (see [Won17c]) and purely technical. The remaining modifications are due to the equation. We see at the first derivative level the decay rates shown are modified from the standard linear rate by $(y^0)^{\gamma}$, while at the second derivative level the decay rates are worse by a factor of $(y^0)^{\gamma+1}$. (For linear waves in $d = 4$, $|L^i L^j \phi|$ should decay like $(y^0)^{-1}$.) This worsened decay is a consequence of the background Υ'' that appears in the equation.

Remark 4.4.6. Notice that we do not make use of fractional Sobolev spaces. In the integer setting, to close the L^2–L^∞ Sobolev estimate, in 4 dimensions we need to take 3 derivatives. Returning to the schematics described in the introduction of this section, we expect the equation satisfied by $L^{\leq 3}\phi$ to have a right hand side growing like $(1+\underline{u})^{-1/2}$. Our bootstrap assumptions (as well as was shown in Table 4.3) indicate, on the other hand, that the inhomogeneity can take a coefficient growing like $(1+\underline{u})^{-1+\epsilon}$ (remember that $\gamma < \frac{1}{3}$ is fixed and arbitrary). This gain of effectively a power of 1/2 is due to our use of an anisotropic energy (see Remark 4.3.15) and that on the support of Υ'' the derivative $\partial_{\underline{u}}$ is well-approximated by a "tangential derivative".

4.4.5 Bootstrap for $d = 3$

We close this section by recording the bootstrap argument for $d = 3$. Here the bootstrap assumptions will be taken to be

$$\mathfrak{E}_k(\tau) \leq \begin{cases} \delta & k = 0, 1 \\ \delta \tau^{k-2+\gamma} & k = 2, 3 \end{cases}. \tag{4.4.15}$$

Here again $\gamma \ll 1$ is fixed to be $< \frac{1}{3}$. The weight bounds are shown in Table 4.4. Arguing similarly to the case $d = 4$ we see that the bootstrap assumptions imply

$$\mathfrak{E}_k(\tau)^2 - \mathfrak{E}_k(2)^2 \lesssim \begin{cases} \delta^3 & k \leq 1 \\ \delta^3 \tau^{2\gamma+2k-4} & k = 2, 3 \end{cases}$$

and hence for sufficiently small δ and ϵ, the bootstrap argument closes and we have global existence.

Table 4.4: ($d = 3$) List of admissible ℓ_0, ℓ_1, ℓ_2 values as well as the corresponding upper bounds for w_{ℓ_0,ℓ_1,ℓ_2}. The value of "—" means any value compatible with the prescribed columns. The shaded rows are those with non-integrable upper bounds for w_{ℓ_0,ℓ_1,ℓ_2}. (Recall that $3\gamma < 1$ by fiat.)

k	ℓ_0	ℓ_1	ℓ_2	$w_{\ell_0,\ell_1,\ell_2}(s) \leq$
≤ 1	—	—	—	$s^{\gamma-2}$
2	2	0	0	$s^{2\gamma-1}$
2	—	—	1	$s^{2\gamma-2}$
2	0	0	2	$s^{3\gamma-3}$
3	3	0	0	$s^{2\gamma+1}$
3	—	—	1	$s^{2\gamma}$
3	—	—	2,3	$s^{3\gamma-1}$

For convenience we record here the corresponding L^∞ decay rates relative to the y coordinates. These can be obtained by applying Proposition 4.3.14 to the bootstrap assumptions above.

$$
\begin{aligned}
&|\phi| \lesssim (y^0)^{-1/2}, && \\
&|L^i\phi| \lesssim (y^0)^{\gamma-1/2}, && |L^iL^j\phi| \lesssim (y_0)^{\gamma+1/2}, \\
&|T\phi| \lesssim (y^0)^{\gamma-1/2}\tau^{-1}, && |TL^i\phi| \lesssim (y^0)^{\gamma+1/2}\tau^{-1}.
\end{aligned}
$$

Remark 4.4.7. An examination of Tables 4.1, 4.2, 4.3, and 4.4 shows that, exactly as discussed in the introduction to this section, the nonlinear terms that cause the main difficulty are those where the commutator vector fields hit *entirely* on the background planewave Υ''. This shows that even if we start by considering initial data with higher degree of regularity, the loss of decay will always appear in the energy $\mathfrak{E}_k$ starting from $k = d-1$.

Remark 4.4.8. In the arguments given above, when d is odd we only commuted with up to d vector fields, and when d is even we used $d+1$ vector fields. It is fairly straightforward to check, in fact, that for initial data of higher regularity, the higher regularity is preserved in the solution. However, for each additional derivative the energy growth speeds up by another factor of τ. So for example, in dimension $d = 3$ the higher energy $\mathfrak{E}_{11}(\tau)$ will have controlled growth like $\tau^{9+\gamma}$ in our bootstrap scheme.

4.5 Commuted equations

We now return to the membrane equation. As discussed in Section 4.2.2, to handle the quasilinearity it is convenient to consider not just (4.2.17) but also the prolonged system (4.2.18) for its first derivatives. As seen in Section 4.4 previously, we will prefer to work with the weighted vector field derivatives $L^i\phi$ instead of the coordinate partials $\partial_\lambda\phi$. In this section we will first write down the corresponding propagation equations for $L^i\phi$.

While the arguments in Section 4.4 sums up neatly our approach toward the semilinear inhomogeneity in the equation, the quasilinear nature of (4.2.17) introduces additional complications. Whereas in the semilinear case the commutation relations $[L^i, \Box_{\mathfrak{m}}] = 0$ hold, in the quasilinear case $[L^i, \Box_g]$ are generally non-vanishing second order differential operators, whose coefficients depend on the unknown ϕ itself. In the second part of this section we perform these basic commutation computations and systematically record the additional terms that arise which would also need to be controlled.

In the final part of this section, we give a statement of our main stability theorem for simple planewave solutions to the membrane equation. We will state and prove our theorem in the most critical case $d = 3$. Returning to the results of Section 4.4, we see that when $d \geq 5$ the solution ϕ to the semilinear equation is such that ϕ and its first order weighted derivatives $L^i\phi, T\phi$ all enjoy pointwise decay at rates identical to the solution to the linear wave equation. For the corresponding quasilinear problem the dynamical metric g also has fast decay toward $\mathfrak{m}$, and the quasilinearity poses almost no additional complications compared to the semilinear case.

As already discussed in the introduction to Section 4.4, in lower spatial dimensions even the semilinearity causes additional difficulties compared to $d \geq 5$; this requires, in particular, that the decay rates of even the lowest order derivatives $L^i\phi$ and $T\phi$ be modified from their expected linear rates. In the quasilinear setting, this causes *additional complications*. In three dimensions, in particular, the appearance of terms of the form

$$\Upsilon''(u)\phi\partial^2_{\underline{u}\underline{u}}\phi$$

in (4.2.18) is potentially troublesome. Based on purely the *linear* peeling estimates, which follows from applying Proposition 4.3.14 to a solution of the linear wave equation, and which would give (on the support of Υ'')

$$|\phi| \lesssim (y^0)^{-1/2}, \quad |\partial_{\underline{u}}\phi| \lesssim (y^0)^{-3/2}, \quad |\partial^2_{\underline{u}\underline{u}}\phi| \lesssim (y^0)^{-5/2},$$

one may naively expect that $\Upsilon''(u)\phi\partial^2_{\underline{u}\underline{u}}\phi$ has similar decay properties as the semilinear nonlinearity $\Upsilon''(u)(\phi_{\underline{u}})^2$ that we already treated. However, if we instead examine the decay rates proven in Section 4.4 (which we should not expect to be better), we have

$$|\phi| \lesssim (y^0)^{-1/2}, \quad |\partial_{\underline{u}}\phi| \lesssim (y^0)^{-3/2+\gamma}, \quad |\partial^2_{\underline{u}\underline{u}}\phi| \lesssim (y^0)^{-3/2+\gamma},$$

making the decay for $\Upsilon''(u)\phi\partial^2_{\underline{u}\underline{u}}\phi$ *slower* by a factor of y^0 compared to the semilinear term.

This potential difficulty is significantly ameliorated in $d \geq 5$; doing a similar analysis using the proven decay rates in Section 4.4 shows that the difference between the quasilinear $\Upsilon''(u)\phi\partial^2_{\underline{u}\underline{u}}\phi$ term and its semilinear counterpart, when $d = 5, 6$ is merely a factor of $\ln(\tau)$ which does not impact the bootstrap argument; and when $d \geq 7$ no difference is present. Hence, both for brevity of presentation and clarity of argument, we shall concentrate the remainder of this chapter on the most difficult case $d = 3$. The higher dimensional cases can all be handled similarly; with the difference being mainly one of bookkeeping.

The overcoming of this potential difficulty with the $\Upsilon''\phi\partial^2_{\underline{u}\underline{u}}\phi$ terms in dimension $d = 3$ relies, unsurprisingly, on the "null structure" of the equation. In Section 4.4 for brevity of argument the derivatives $L^1\phi$ and $L^i\phi$ for $i = 2, \ldots, d$ are estimated isotropically. However, the equations that they satisfy are not the same: recalling that the worst term of the inhomogeneity arises from when the weighted vector fields hit the background Υ'', we expect

$$\Box L^1\phi \approx (L^1\Upsilon'')(\phi_{\underline{u}})^2$$

in the semilinear argument. However, a direct computation shows that

$$L^1\Upsilon'' = u\Upsilon'''$$

is again a smooth function with compact support in u. In particular, while for $i = 2, \ldots, d$ we have the *growing weights* as described in Lemma 4.3.13, this loss *is not seen by pure L^1 derivatives*. Therefore we expect $L^1\phi$ to actually enjoy better decay compared to $L^i\phi$ for $i \neq 1$. Finally, returning to the difficult term $\phi_{\underline{u}\underline{u}}$, we see that the $\partial_{\underline{u}}$ derivative lies in the span of T and L^1 (see also (4.5.8) and Remark 4.5.3); hence we will expect that $\partial^2_{\underline{u}\underline{u}}\phi$ to decay *faster* than the generic tangential second derivative, allowing us to eventually close our estimates.

Remark 4.5.1. In $d = 2$ this observation is in fact enough to allow us to close the energy estimate for the *semilinear* model. However, additional difficulties come up in the analysis of the full *quasilinear* problem that cannot be treated using only this method, hence we omit its discussion below. For the semilinear problem (4.4.1), let us denote by $\mathfrak{E}_k$ the k-th order energies for ϕ, and $\mathfrak{F}_k$ the k-th order energies for $L^1\phi$ (analogously to how we proceed in Section 4.6 below for the quasilinear problem in $d = 3$). This way of treating the equations for ϕ and $L^1\phi$ separately allows us to close the global-existence bootstrap in a manner similar to that described in Section 4.4 with the energy bounds

$$\begin{aligned} \mathfrak{E}_0, \mathfrak{F}_0 &\lesssim \delta, \\ \mathfrak{E}_1, \mathfrak{F}_1 &\lesssim \delta\tau^{\gamma}, \\ \mathfrak{E}_2, \mathfrak{F}_2 &\lesssim \delta\tau^{1+\gamma}, \\ \mathfrak{E}_3 &\lesssim \delta\tau^{2+\gamma}. \end{aligned}$$

For the quasilinear problem, this scheme breaks down when dealing with the $TT\phi$ derivatives that crop up.

In dimension $d \geq 3$, the cubic and higher nonlinearities are essentially harmless, even with the slightly reduced decay rates. (In the linear case the terms placed in L^∞ combine to decay at least as fast $(y^0)^{-3/2}$; a loss of $\gamma < \frac{1}{3}$ can be easily absorbed.) This fact allows us to essentially ignore all "null structure" when handling the cubic and higher order terms, which allows us to significantly simplify the bookkeeping involved.

4.5.1 The perturbed system, restated

Our goal this section is to derive the evolution equations for $L^i\phi$. Some of the computations are lengthy and not entirely transparent: they are recorded in Appendix A.3.1. We start with (4.2.17) which we re-write as

$$\sqrt{|g|}\partial_\mu \frac{\mathring{g}^{\mu\nu}\partial_\nu\phi}{\sqrt{|g|}} = \Upsilon''(\phi_{\underline{u}})^2.$$

We expand the left hand side as

$$\begin{aligned}\Box_{\mathfrak{m}}\phi + 2\partial_{\underline{u}}\Big(\phi\Upsilon''\partial_{\underline{u}}\phi\Big) + \sqrt{|g|}\mathring{g}(\mathrm{d}\phi, \mathrm{d}|g|^{-1/2}) \\ = \Box_{\mathfrak{m}}\phi + 2\partial_{\underline{u}}\Big(\phi\Upsilon''\partial_{\underline{u}}\phi\Big) - \frac{1}{2|g|}\mathring{g}\Big(\mathrm{d}\phi, \mathrm{d}\Big(\mathring{g}(\mathrm{d}\phi, \mathrm{d}\phi)\Big)\Big).\end{aligned}$$

Notice, on the other hand, that

$$\begin{aligned}\Box_g\psi = \Box_{\mathfrak{m}}\psi + 2\partial_{\underline{u}}\Big(\phi\Upsilon''\partial_{\underline{u}}\psi\Big) - \frac{1}{|g|}\mathring{g}\Big(\mathrm{d}\phi, \mathrm{d}\Big(\mathring{g}(\mathrm{d}\phi, \mathrm{d}\psi)\Big)\Big) \\ - \frac{1}{|g|}\Upsilon''(\phi_{\underline{u}})^2 \cdot \mathring{g}(\mathrm{d}\phi, \mathrm{d}\psi) + \frac{1}{2|g|}\mathring{g}(\mathrm{d}|g|, \mathrm{d}\psi). \quad (4.5.1)\end{aligned}$$

Together this implies that, if X is a Killing vector field of the Minkowski metric $\mathfrak{m}$, that

$$\begin{aligned}\Box_g X\phi = &\boxed{X\big(\Upsilon''(\phi_{\underline{u}})^2\big)} - \frac{1}{|g|}\Upsilon''(\phi_{\underline{u}})^2\mathring{g}(\mathrm{d}\phi, \mathrm{d}(X\phi)) + \frac{1}{|g|}\mathring{g}(\mathrm{d}|g|, \mathrm{d}(X\phi)) \\ &- \boxed{2[X,\partial_{\underline{u}}](\phi\Upsilon''\phi_{\underline{u}})} - \boxed{2\partial_{\underline{u}}(X(\phi\Upsilon'')\phi_{\underline{u}})} - \boxed{2\partial_{\underline{u}}(\phi\Upsilon''[X,\partial_{\underline{u}}]\phi)} \\ &- \frac{1}{2|g|^2}X(|g|)\mathring{g}(\mathrm{d}\phi, \mathrm{d}|g|) + \frac{1}{2|g|}\mathcal{L}_X(\mathring{g}^{-1})(\mathrm{d}\phi, \mathrm{d}|g|) \\ &+ \frac{1}{2|g|}\mathring{g}\Big(\mathrm{d}\phi, \mathrm{d}\big(\mathcal{L}_X(\mathring{g}^{-1})(\mathrm{d}\phi, \mathrm{d}\phi)\big)\Big). \end{aligned} \tag{4.5.2}$$

Here, $\mathcal{L}_X\mathring{g}^{-1}$ is the Lie derivative of the inverse metric $\mathring{g}^{-1}$ by the vector field X. It can be given as

$$\mathcal{L}_X\mathring{g}^{-1} = 2X(\phi\Upsilon'')\partial_{\underline{u}}\otimes\partial_{\underline{u}} + 2\phi\Upsilon''[X,\partial_{\underline{u}}]\otimes\partial_{\underline{u}} + 2\phi\Upsilon''\partial_{\underline{u}}\otimes[X,\partial_{\underline{u}}]. \tag{4.5.3}$$

The boxed terms in (4.5.2) are those with *quadratic nonlinearity* and are the ones for which the null structure play an important role. The remaining terms on the right hand side all have cubic or higher nonlinearities, and will be treated more roughly in the estimates.

Later on we will take X to be one of L^i; we can compute the commutators (see (4.3.6) and (4.3.7) for definitions)

$$[L^1,\partial_{\underline{u}}] = -\partial_{\underline{u}}; \tag{4.5.4}$$

$$[L^i,\partial_{\underline{u}}] = -\frac{1}{\sqrt{2}}\frac{1}{y^0}(L^i - y^i T), \qquad i\in\{2,\dots,d\}. \tag{4.5.5}$$

For convenience, we will introduce the following *schematic notations*.

Notation 4.5.2. First, in view of Lemma 4.3.13, we will denote by $\mathscr{P}_m$ any finite sum of terms of the form

$$\Big(\text{Polynomial in } \{\underline{u},\hat{x}\}\Big)\cdot\Big(\text{Compactly supported smooth function of } u\Big) \tag{4.5.6}$$

such that on $\mathcal{I}^+$ it is bounded by $(1+\underline{u})^{m/2}$. Our assumptions imply $\Upsilon'' = \mathscr{P}_0$. The computations surrounding the proof of Lemma 4.3.13 imply that

$$T\mathscr{P}_m = \mathscr{P}_m, \qquad L^1\mathscr{P}_m = \mathscr{P}_m, \qquad L^i\mathscr{P}_m = \mathscr{P}_{m+1} \text{ for } i\in\{2,\dots,d\}. \tag{4.5.7}$$

We will denote by $\mathscr{W}_m$ any element of $\mathfrak{W}_m$.

With these notations, we can rewrite schematically

$$\mathscr{P}_m \partial_{\underline{u}} = \mathscr{W}_1 \mathscr{P}_m \Big(L^1 + T \Big). \tag{4.5.8}$$

Remark 4.5.3 (Vestige of null condition)**.** As discussed in Remark 4.2.3, the presence of the Υ'' factor in $\Upsilon''(\phi_{\underline{u}})^2$ helps to ameliorate the resonant interaction. This improvement is a vestige of the null condition of the original membrane equation. In our reformulation here, this improvement is captured in (4.5.8) above. Observe that a generic coordinate derivative ∂_u, $\partial_{\underline{u}}$, or $\partial_{\hat{x}}$ can be written only as an element of the commutator algebra $\mathfrak{A}_1$, which means that the *transversal* factor T is not accompanied by a decaying weight. From this one can see that quadratic terms of the form $(T\phi)^2$ will serve as a severe obstacle to global existence. In our setting, however, the $\mathscr{P}_0$ weight Υ'' provides a spatial localization and gives an *anomalous weighting*: the term $\mathscr{W}_1 T \in \mathfrak{A}_2$ and has *improved decay* and this improvement is, fundamentally, what allows our argument to close in this chapter.

Notation 4.5.4. We will frequently denote by $\mathscr{B}_w^{k,s}$ an element of $\mathfrak{B}_{w'}^{k,s}$ with $w' \geq w$. When $|g|$ appears in a higher order term, if is often sufficient to control it as

$$|g| = 1 + (\mathscr{B}_1^{1,1}\phi)^2(1 + \mathscr{P}_0\phi), \tag{4.5.9}$$

and similarly we can write

$$\mathring{g}(\mathrm{d}\varphi, \mathrm{d}\psi) = (\mathscr{B}_1^{1,1}\varphi)(\mathscr{B}_1^{1,1}\psi)(1 + \mathscr{P}_0\phi). \tag{4.5.10}$$

Remark 4.5.5. Observe that in (4.5.2), the inhomogeneity depends on *up to second order derivatives of* ϕ. If we decompose nonlinearities, the second order derivatives that appear are *generic*, in the sense that derivatives with principal parts $TT\phi$, $TL^i\phi$, and $L^iL^j\phi$ all appear. (Note that $\{T, L^i\}$ span the tangent space $\mathbb{R}^{1,d}$.) To control $TL^i\phi$ and $L^iL^j\phi$ in L^∞, by Proposition 4.3.14 it suffices to control the energies of $L^\alpha\phi$. The term $TT\phi$, however, is *not* controlled by these energies. There are two approaches to address this. First is to enlarge the set of commutators required; instead of only commuting with the boosts

L^{α}, one can commute with also the T vector field. Checking the commutator relations, to close this argument one would have to commute with all differential operators of the form $L^{\alpha}T^{k}$ where $|\alpha|+k$ is bounded by some k_0. For our problem, it appears slightly simpler computationally to take the second (essentially equivalent) alternative. By decomposing $\Box_g$ we can solve (4.2.17) for $TT\phi$ in terms of $TL^i\phi$ and $L^iL^j\phi$ and lower order derivatives. This implies $TTL^{\beta}\phi$ can be estimated in terms of $TL^{\gamma}\phi$ and $L^{\alpha}\phi$ where $|\gamma|\leq|\beta|+1$ and $|\alpha|\leq|\beta|+2$. See Appendix A.3.1 for the details of this computation.

Notation 4.5.6. We will denote by $\mathscr{G}=\mathscr{G}(\phi\mathscr{P}_0,\mathscr{B}_0^{1,0}\phi,\mathscr{B}_1^{1,1}\phi)$ an arbitrary smooth function of its arguments. In particular, $|g|=\mathscr{G}$ in this notation, as well as $|g|^{-1}=\mathscr{G}$ when the $\phi,\mathscr{B}_0^{1,0}\phi$, and $\mathscr{B}_1^{1,1}\phi$ are all sufficiently small.

It is convenient to simplify (4.5.2) a bit more.

With the aid of these schematic notations, we find that $L^1\phi$ satisfies

$$\begin{aligned}\Box_g L^1\phi = \mathscr{P}_0\mathscr{W}_2\cdot\Big[&(L^1\phi+T\phi)^2+T\phi(L^1L^1\phi+TL^1\phi)\\ &+\phi(L^1\phi+T\phi)+(\phi+L^1\phi)(L^1L^1\phi+TL^1\phi+TT\phi)\Big]\\ &+\mathscr{G}(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{1,1}L^1\phi)(\mathscr{B}_2^{2,2}\phi)+\mathscr{G}\mathscr{P}_0(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_2^{2,2}\phi)\\ &+\mathscr{G}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^3(\mathscr{B}_1^{2,1}\phi)+\mathscr{G}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^5(\mathscr{B}_2^{2,2}\phi)+\mathscr{G}\mathscr{P}_0(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^4\\ &+\mathscr{G}\phi\mathscr{W}_1\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{2,1}\phi)+\mathscr{G}\phi^2\mathscr{W}_2\mathscr{P}_2(\mathscr{B}_1^{1,1}\phi)^4(\mathscr{B}_1^{2,1}\phi)\\ &+\mathscr{G}\phi\mathscr{W}_1\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^5(\mathscr{B}_2^{2,2}\phi)+\mathscr{G}\mathscr{W}_1\mathscr{P}_1(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^3\end{aligned}\tag{4.5.11}$$

The first brackets capturing all the quadratic nonlinearities and the cubic and higher nonlinearities are described schematically after. For $i \neq 1$, the term $L^i\phi$ satisfies the equation

$$\begin{aligned}\Box_g L^i\phi = \mathscr{P}_0\mathscr{W}_1(\phi + L^1\phi + T\phi)(\mathscr{B}_1^{1,1}L^1\phi + \mathscr{B}_2^{2,2}\phi + \mathscr{B}_1^{1,1}\phi)\\ + \mathscr{W}_2(\mathscr{P}_0 L^i\phi + \mathscr{P}_1\phi)(L^1L^1\phi + TL^1\phi + TT\phi + L^1\phi + T\phi)\\ + \mathscr{P}_1\mathscr{W}_2(\phi + L^1\phi + T\phi)(L^1\phi + T\phi)\\ + \mathscr{G}(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_2^{2,2}\phi) + \mathscr{G}\mathscr{P}_0(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_2^{2,2}\phi)\\ + \mathscr{G}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^3(\mathscr{B}_1^{2,1}\phi) + \mathscr{G}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^5(\mathscr{B}_2^{2,2}\phi) + \mathscr{G}\mathscr{P}_0(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^4\\ + \mathscr{G}\phi\mathscr{W}_1\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{2,1}\phi) + \mathscr{G}\phi^2\mathscr{W}_2\mathscr{P}_2(\mathscr{B}_1^{1,1}\phi)^4(\mathscr{B}_1^{2,1}\phi)\\ + \mathscr{G}\phi\mathscr{W}_1\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^5(\mathscr{B}_2^{2,2}\phi) + \mathscr{G}\mathscr{W}_1\mathscr{P}_1(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^3\\ + \mathscr{G}\phi\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_2^{2,2}\phi) + \mathscr{G}\phi\mathscr{W}_1\mathscr{P}_2(\mathscr{B}_1^{1,1}\phi)^3 + \mathscr{G}\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^4. \quad (4.5.12)\end{aligned}$$

Note that the cubic and higher-order terms are schematically represented largely in the same way, with the main differences coming in the quadratic terms. The key observation, as already mentioned in the introduction to this section, is that the quadratic terms in the equation for $L^1\phi$ do not see the growing weight term, and therefore behaves like ϕ instead of a generic $L\phi$ term. This improvement then also propagates into the analysis of the quadratic terms of equation (4.5.12) of the general L derivatives.

For convenience, we record (4.2.17) here in the schematic notation.

$$\Box_g\phi = \mathscr{G}\mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)^2. \quad (4.5.13)$$

4.5.2 Commutator relations

To use the vector field method, we will be commuting our equations with the L^i derivatives. More precisely, we study the wave equations satisfied by $\mathscr{B}_0^{k,0}(L^1\phi, L^i\phi)$ by writing

$$\Box_g(\mathscr{B}_0^{k,0}L\phi) = \mathscr{B}_0^{k,0}(\Box_g L\phi) + [\mathscr{B}_0^{k,0}, \Box_g](L\phi).$$

Note that after applying (4.5.11) and (4.5.12) the right-side does not contain principal terms. Differentiation of the schematic relations in (4.5.11), (4.5.12), and (4.5.13) are straightforward. To implement our strategy, we need to compute the commutators $[X, \Box_g]$ acting on a smooth scalar ψ, where $X = L^1$ or L^i. We merely record the results here, and defer the actual computation to A.3.2.

$$
\begin{aligned}
[X, \Box_g]\psi = {} & \mathscr{P}_0\mathscr{W}_1(\mathscr{B}_1^{1,1}\phi)(L^1\psi + T\psi) + \mathscr{P}_0\mathscr{W}_1(\phi + L^1\phi + T\phi)(\mathscr{B}_1^{1,1}\psi) \\
& + \mathscr{P}_0\mathscr{W}_1(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}L^1\psi + \mathscr{B}_1^{1,1}T\psi) + \mathscr{P}_0\mathscr{W}_2(\mathscr{B}_0^{1,0}L^1\phi + \mathscr{B}_0^{1,0}T\phi)(L^1\psi + T\psi) \\
& + \mathscr{P}_1\mathscr{W}_2(\phi + L^1\phi + T\phi)(L^1\psi + T\psi) + \mathscr{P}_1\mathscr{W}_2\phi(L^1L^1\psi + TL^1\psi + TT\psi) \\
& + (X\mathscr{G})\cdot\Big[(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_2^{2,2}\phi)(\mathscr{B}_1^{1,1}\psi) + (\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_2^{2,2}\psi) \\
& + \mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^3(\mathscr{B}_1^{1,1}\psi) + \mathscr{P}_1\mathscr{W}_1\phi(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{1,1}\psi)\Big] \\
& + \mathscr{G}\cdot\Big[(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_2^{2,2}\phi)(\mathscr{B}_1^{1,1}\psi) + (\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_2^{3,2}\phi)(\mathscr{B}_1^{1,1}\psi) \\
& + (\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_2^{2,2}\psi) + \mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_1^{1,1}\psi) \\
& + \mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^3(\mathscr{B}_1^{1,1}\psi) + \mathscr{P}_1\mathscr{W}_1\phi(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_1^{1,1}\psi) \\
& + \mathscr{P}_1\mathscr{W}_1(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{1,1}\psi) + \mathscr{P}_2\mathscr{W}_1\phi(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{1,1}\phi)\Big]. \qquad (4.5.14)
\end{aligned}
$$

Notice that the quadratic terms (linear in both ϕ and ψ) are listed explicitly, as we expect to need to use the null structure to extract sufficient decay. The cubic and higher terms (which are at least quadratic in the background ϕ), are listed purely schematically.

Remark 4.5.7. Now and in the sequel, HO_1 constitutes the cubic and higher order terms that arise in the right hand side of (4.5.11), see also Appendix A.3.1.4. Similarly, HO_i for equation (4.5.12), see also Appendix A.3.1.5. A key thing to note about the commutator relation (4.5.14) is that, with $\psi = L^\alpha L^1\phi$ for some multi-index α, every *cubic and higher* term that appears in the schematic decomposition above can be obtained, schematically, as a term that appears in an $L^{\leq|\alpha|+1}$ derivative of HO_1. And similarly with $\psi = L^\alpha\phi$ every cubic and higher term in the schematic decomposition is a term that appears in an $L^{\leq|\alpha|}$ derivative of HO_i. (The only difference being our schematic treatment of the purely cubic term; see Remark A.3.2.). Thus we will not separately treat the cubic and higher terms that arise from the commutator in our analyses later, and absorb it as part of the general discussion of higher order terms.

Similarly, with $\psi = L^\alpha \phi$ all the quadratic terms that appear in (4.5.14) can be obtained from $L^{\leq|\alpha|}$ derivatives hitting on QN_i, which are defined as the quadratic inhomogeneity of (4.5.12). However as we can see in the case $\psi = L^\alpha L^1$, the final quadratic commutator term of the form $\mathscr{P}_1 \mathscr{W}_2 \phi(L^1 L^1 \psi + T L^1 \psi + T T \psi)$ cannot be obtained as an $L^{\leq|\alpha|+1}$ derivative of QN_1, which are defined as the quadratic homogeneity of (4.5.11) (notice the differing weights $\mathscr{P}_0$ and $\mathscr{P}_1$). These turn out to be the most delicate terms in the analysis, and in Section 4.7.2.4 will be the main terms to saturate the polynomial growth in the energy estimates.

4.5.3 Statement of the main theorem

Our main theorem asserts that when the initial planewave Υ has bounded width, then this travelling wave solution is stable under small compactly supported perturbations. By rescaling and translating we can assume the perturbation is supported in the unit ball $B(0,1) \subset \mathbb{R}^3$ on the spatial slice $\{y^0 = 2\}$.

Theorem 4.5.8. *Let $d = 3$ and assume $\Upsilon(u)$ is such that Υ'' has compact support in u. Consider the initial value problem for* (4.2.17)*, where the dynamical metric is given by* (4.2.9)*. We assume the initial data is prescribed on the spatial slice $\{y^0 = 2\}$ by*

$$\phi|_{y^0=2} = \phi_0, \quad \partial_{y^0}\phi|_{y^0=2} = \phi_1,$$

where $\phi_0, \phi_1 \in C_0^\infty(B(0,1))$. Then for any $\gamma > 0$ there exists some $\epsilon_0 > 0$ (which we allow to depend on Υ and on γ) such that whenever

$$\|\phi_0\|_{H^5} + \|\phi_1\|_{H^4} \leq \epsilon_0$$

the solution ϕ exists for all time $y^0 \geq 2$. Furthermore, we have the following uniform bounds *on the solution and its derivatives:*

$$
\begin{gathered}
|\phi| + |L^1\phi| \lesssim (y^0)^{-1/2}, \\
|T\phi| + |TL^1\phi| \lesssim \tau^{\gamma-1}(y^0)^{-1/2}, \\
|\mathscr{B}_0^{1,0}\phi| + |\mathscr{B}_0^{1,0}L^1\phi| + |T\mathscr{B}_0^{1,0}\phi| + |TT\phi| \lesssim \tau^{\gamma}(y^0)^{-1/2}, \\
|\mathscr{B}_0^{2,0}\phi| + |\mathscr{B}_1^{3,1}\phi| \lesssim \tau^{1+\gamma}(y^0)^{-1/2}.
\end{gathered}
$$

Remark 4.5.9. Observe that in particular, the coordinate derivatives (with respect to y) up to second order all decay uniformly as $y^0 \nearrow \infty$. As will be clear from the proof, if the initial data has higher regularity the regularity persists for the solution. This can be extended to show that (the details of the proof we omit here) that arbitrary order coordinate derivatives of the solution decay uniformly like $(y^0)^{-1/2+\gamma}$. Peeling, however, doesn't hold to arbitrary orders, unlike the case of the linear wave. If we denote by $\bar{\partial}$ a derivative that is tangential to out-going Minkowski light-cones, our results are only compatible with these outgoing tangential derivatives $\bar{\partial}^{\beta}\phi$ being uniformly bounded by $(y^0)^{-3/2+\gamma}$ *for all orders* $|\beta| \geq 2$.

4.6 Energy quantities and bootstrap assumptions

The remainder of this chapter is devoted to proving Theorem 4.5.8. In view of the robust local existence theory for nonlinear wave equations, the strategy we will take is that of a standard bootstrap argument. In this section we will set the notations for the basic energy quantities and perform some preliminary analyses on them, having also introduced the main bootstrap assumptions.

4.6.1 The energy quantities defined; bootstrap assumptions

Recall from (4.3.14) the energy quantity

$$\mathcal{E}_\tau[\psi;g]^2 = 2\int\limits_{\Sigma_\tau} \frac{1}{\sqrt{|g(\mathrm{d}\tau,\mathrm{d}\tau)|}}\mathcal{Q}[\psi;g](T,(-\mathrm{d}\tau)^{g\sharp})\ \mathrm{dvol}_{h_\tau}$$

which satisfies the basic energy inequality (4.3.16) for $\tau_0 < \tau_1$

$$\mathcal{E}_{\tau_1}[\psi;g]^2 \leq \mathcal{E}_{\tau_0}[\psi;g]^2 + \iint\limits_{\tau\in[\tau_0,\tau_1]} \left|\mathcal{Q}[\psi;g] :_g \mathcal{L}_T g\right| + 2\left|\Box_g\psi\cdot T(\psi)\right|\ \mathrm{dvol}_g. \tag{4.6.1}$$

Here ψ will stand for some higher L derivative of the solution ϕ. One difference between our quasilinear setting and the semilinear model treated in Section 4.4 is the presence of the first integrand in the energy inequality. In the semilinear case $\mathcal{L}_T\mathfrak{m} = 0$. The analysis of the second integrand will occupy Section 4.7, using the equations (4.5.13), (4.5.11), (4.5.12); we treat the first integrand here.

The integrand can be expanded as

$$\mathcal{Q}[\psi;g] :_g \mathcal{L}_T g = (\mathcal{L}_T g^{-1})(\mathrm{d}\psi,\mathrm{d}\psi) - \frac{1}{2}g^{-1}(\mathrm{d}\psi,\mathrm{d}\psi)\cdot g :_g \mathcal{L}_T g.$$

We primarily care about terms that are linear in ϕ: the terms with higher order dependence on ϕ we expect to behave better and will estimate very roughly. With that, and (4.2.14) in mind, schematically

$$\begin{aligned}(\mathcal{L}_T g^{-1})(\mathrm{d}\psi,\mathrm{d}\psi) &= (\phi + T\phi)\mathscr{P}_0\mathscr{W}_2(L^1\psi + T\psi)^2 \\ &\quad + \mathscr{G}\left[(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_2^{2,2}\phi) + (\mathscr{B}_1^{1,1}\phi)^2(1 + T\phi\mathscr{P}_0)\right](\mathscr{B}_1^{1,1}\psi)^2.\end{aligned}$$

And we also have schematically, by (4.2.9), that

$$\begin{aligned} g :_g \mathcal{L}_T g &= g^{-1}(\mathrm{d}\phi,\mathrm{d}T\phi) + g^{-1}(\mathrm{d}u,\mathrm{d}u)(\phi + T\phi)\mathscr{P}_0 \\ &= \mathscr{G}(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_2^{2,2}\phi) + \mathscr{G}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^2(\phi + T\phi).\end{aligned}$$

Therefore we can conclude that schematically

$$\begin{aligned}\mathcal{Q}[\psi;g] :_g \mathcal{L}_T g &= (\phi + T\phi)\mathscr{P}_0\mathscr{W}_2(L^1\psi + T\psi)^2 \\ &\quad + \mathscr{G}\left[(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_2^{2,2}\phi) + (\mathscr{B}_1^{1,1}\phi)^2(1 + T\phi\mathscr{P}_0)\right](\mathscr{B}_1^{1,1}\psi)^2. \end{aligned}\tag{4.6.2}$$

We will return to estimating this term in Section 4.6.5

For convenience, for $\tau \geq 2$ and k a non-negative integer, we will denote by

$$\mathfrak{E}_k(\tau) \stackrel{\text{def}}{=} \sup_{\sigma \in [2,\tau]} \mathcal{E}_\sigma[L^{\leq k}\phi; g], \tag{4.6.3}$$

$$\mathfrak{F}_k(\tau) \stackrel{\text{def}}{=} \sup_{\sigma \in [2,\tau]} \mathcal{E}_\sigma[L^{\leq k}L^1\phi; g]. \tag{4.6.4}$$

We will make the following *initial data assumption*:

$$\mathfrak{E}_4(2) + \mathfrak{F}_3(2) \leq \epsilon \tag{A$_{\text{ID}}$}$$

for some $\epsilon \geq \epsilon_0$. We can make this assumption as by the standard local-existence argument for nonlinear wave equations, with the assumptions in Theorem 4.5.8, for sufficiently small ϵ_0 the solution necessarily exists up to Σ_2. The continuity of the energy norms on initial data implies that as $\epsilon_0 \to 0$ the quantity $\mathfrak{E}_4(2) + \mathfrak{F}_3(2) \to 0$ also.

As is typical of bootstrap arguments, we will assume there is some $T > 2$ such that for every $\tau \in [2, T]$ the following *bootstrap assumptions* hold. We need three parameters: $\delta_0 > 0$ whose size will be fixed in Section 4.6.3 and considered constant afterwards; $\delta \in (0, \delta_0)$ which is a smallness parameter we will adjust to close the bootstrap. Without loss of generality we will assume $\gamma \in (0, 1/4)$ is fixed throughout the argument. Our goal, as usual, is to demonstrate that the bootstrap assumptions below leads to improved versions of themselves, when δ and ϵ are taken to be sufficiently small. This then implies by standard continuity argument that the assumptions in fact hold for all times $\tau > 2$ and we obtain global existence.

Our bootstrap assumptions are: First, along Σ_τ we have the uniform bounds

$$\begin{cases} |\phi| \leq \delta_0 (y^0)^{-1/2}; \\ |L^1\phi| \leq \delta_0 (y^0)^{-1/2}; \\ |L^i\phi| \leq \delta_0 (y^0)^{-1/2}\tau^\gamma; \\ |T\phi| \leq \delta_0 (y^0)^{-1/2}\tau^{\gamma-1}. \end{cases} \tag{BA$_\infty$}$$

Second, we assume that

$$\begin{cases} \mathfrak{E}_1(\tau)+\mathfrak{F}_1(\tau)\le\delta; \\ \mathfrak{E}_2(\tau)+\mathfrak{F}_2(\tau)\le\delta\tau^{\gamma}; \\ \mathfrak{E}_3(\tau)+\mathfrak{F}_3(\tau)\le\delta\tau^{1+\gamma}; \\ \qquad\quad \mathfrak{E}_4(\tau)\le\delta\tau^{2+\gamma}. \end{cases} \tag{BA$_2$}$$

4.6.2 Inequalities on that we use frequently

In the subsequent analysis, we will freely use the control of y^0, $\cosh(\rho)$, and $\underline{u}$ afforded by Lemma 4.6.1. As we will see, these estimates will be an important tool to obtain coercive control (with respect to $\mathfrak{E}_k$, $\mathfrak{F}_k$) of terms that arise in the energy estimates. They also have important consequences when used concurrently with the bootstrap assumptions, see, for instance, Proposition 4.6.2.

Lemma 4.6.1. *The following estimates hold on $\mathcal{I}^+\cap\{\operatorname{supp}\phi\}\cap\{\operatorname{supp}\mathscr{P}_0\}$*

$$\underline{u}\approx y^0 \tag{4.6.5}$$

$$y^0\approx\tau^2 \tag{4.6.6}$$

$$\cosh(\rho)\approx\tau. \tag{4.6.7}$$

Proof. Using $y^0=(u+\underline{u})/\sqrt{2}$, (4.6.5) follows because $\mathscr{P}_0$ has compact support in u. Under the assumptions of the initial data in Theorem 4.5.8, finite speed of propagation implies that

$$\sqrt{|y^1|^2+|y^2|^2+|y^3|^2}\le|y^0-2|+1=y^0-1$$

on the support of ϕ. Since $\tau^2=2u\underline{u}-|\hat{x}|^2=(y^0)^2-(y^1)^2-(y^2)^2-(y^3)^2$, the previous inequality reads $2y_0\le\tau^2+1$ and hence $y^0\lesssim\tau^2$ because $\tau\ge2$. Since $2u\underline{u}\ge\tau^2$ on $\mathcal{I}^+$ (see Section 4.3.1),

$$\tau^2\le2u\underline{u}\lesssim\underline{u}\approx y^0$$

by appealing to the support of $\mathscr{P}_0$. We have then proved (4.6.6). Finally, (4.6.7) follows from the identity $\tau\cosh(\rho) = y^0$ and (4.6.6). □

4.6.3 Some first consequences of (BA$_\infty$)

The assumptions (BA$_\infty$) are not strictly speaking necessary; its presence however helps jump-start basic geometric comparisons that simplifies especially the energy comparisons to be taken in the next subsection.

Proposition 4.6.2. *The assumptions* (BA$_\infty$) *imply*

$$|\mathscr{P}_0\phi| \lesssim \delta_0\tau^{-1};$$
$$|\mathscr{B}_0^{1,0}\phi| \lesssim \delta_0(y^0)^{-1/2}\tau^{\gamma};$$
$$|\mathscr{B}_1^{1,1}\phi| \lesssim \delta_0(y^0)^{-1/2}\tau^{\gamma-1}.$$

And hence

$$|\mathscr{G}| \lesssim 1.$$

Proof. The estimates on $|\mathscr{B}_0^{1,0}\phi|$ and $|\mathscr{B}_1^{1,1}\phi|$ are trivial using the assumptions, together with the fact that $y^0 \geq \tau$ by definition. The estimate on $|\mathscr{P}_0\phi|$ follows from the bootstrap assumption and the estimate (4.6.6) in Lemma 4.6.1. Finally, as $\gamma < 1/2$ by assumption, we see that the three $\mathscr{B}_0^{1,0}$, $\mathscr{B}_1^{1,1}$, and $\mathscr{P}_0\phi$ all have global uniform bounds, therefore we must also have global uniform bounds on the arbitrary smooth functions $\mathscr{G}$. □

Proposition 4.6.3 (Geometric consequences)**.** *The assumptions* (BA$_\infty$) *implies, when δ_0 is sufficiently small, that*

1. *The Jacobian determinant $\frac{1}{2} \leq |g| \leq 2$.*

2. *The hyperboloids Σ_τ are space-like relative to g; in fact $g^{-1}(\mathrm{d}\tau, \mathrm{d}\tau) = -1 + O(\delta_0\tau^{-5/2})$.*

3. *The volume forms $\mathrm{dvol}_{\eta_\tau}$ and dvol_{h_τ} are uniformly comparable.*

4. *The quantity* $\mathfrak{c}_{TT}$ *from* (A.3.3) *is comparable to* $\tau^2/(y^0)^2$.

Proof. The first claim follows from the fact that

$$|g| = 1 + \mathring{g}^{-1}(\mathrm{d}\phi, \mathrm{d}\phi) = 1 + \mathscr{G}(\mathscr{B}_1^{1,1}\phi)^2.$$

For the second claim it suffices to prove bounds on $g^{-1}(\mathrm{d}\tau, \mathrm{d}\tau)$. From (4.2.14) we have that

$$g^{-1}(\mathrm{d}\tau, \mathrm{d}\tau) = \underbrace{\mathfrak{m}^{-1}(\mathrm{d}\tau, \mathrm{d}\tau)}_{=-1} + 2\phi\Upsilon''(\partial_{\underline{u}}\tau)^2 - \frac{1}{|g|}(\mathring{g}^{-1}(\mathrm{d}\tau, \mathrm{d}\phi))^2.$$

By definition

$$\partial_{\underline{u}}\tau = \frac{u}{\tau}$$

and since Υ'' has compact support in u the middle term $\lesssim \delta_0\tau^{-3}$. For the final term we have schematically

$$\begin{aligned}\mathring{g}^{-1}(\mathrm{d}\tau, \mathrm{d}\phi) &= \mathfrak{m}^{-1}(\mathrm{d}\tau, \mathrm{d}\phi) + \phi\Upsilon''\partial_{\underline{u}}\tau\partial_{\underline{u}}\phi \\ &= \frac{\tau}{y^0}T\phi + \sum_i \frac{y^i}{y^0\tau}L^i\phi + \phi\Upsilon''\frac{u}{\tau}\frac{1}{y^0}(L^1\phi + T\phi)\end{aligned}$$

and so we see $|\mathring{g}^{-1}(\mathrm{d}\tau, \mathrm{d}\phi)| \lesssim \delta_0\tau^{\gamma}(y^0)^{-3/2}$. This implies that the final term decays at least as fast as $(\delta_0)^2\tau^{2\gamma}(y^0)^{-3}$ and hence for sufficiently small δ_0 we have the desired bounds.

For the third claim we first examine (4.3.4), as the induced volume form on Σ_τ is given by the interior product of the space-time volume form with the unit normal. By the explicit form of g and the pointwise bounds of Proposition 4.6.3, it suffices that $(\mathrm{d}\tau)^\sharp - (\mathrm{d}\tau)^{g\sharp}/\sqrt{|g(\mathrm{d}\tau, \mathrm{d}\tau)|}$ is bounded when measured by $\mathfrak{m}$. Due to the above bound on $g(\mathrm{d}\tau, \mathrm{d}\tau)$, it suffices to control

$$(\mathrm{d}\tau)^{g\sharp} - (\mathrm{d}\tau)^\sharp = 2\frac{u}{\tau}\phi\Upsilon''\partial_{\underline{u}} - \mathring{g}^{-1}(\mathrm{d}\phi, \mathrm{d}\tau)\partial^{g\sharp}\phi.$$

In terms of the coordinate basis ∂_{y^μ}, the coefficients of the right hand side can be worked out to be bounded by

$$\delta_0\tau^{-2} + (\delta_0)^2\tau^{2\gamma-1}(y^0)^{-2}.$$

This implies the desired conclusion.

The fourth and final claim follows immediately from the definition of (A.3.3). □

For conducting the estimates, we will frequently need to swap between the quantities $\mathcal{E}_\tau[\psi;\mathfrak{m}]^2$, $\mathcal{E}_\tau[\psi;g]^2$, and

$$\int_{\Sigma_\tau} \frac{1}{\tau^2\cosh(\rho)}\sum |L^i\psi|^2 + \frac{1}{\cosh(\rho)}|T\psi|^2 \,\mathrm{dvol}_{h_\tau}.$$

These three quantities turns out to be comparable if we assume (BA_∞) holds with δ_0 sufficiently small.

Proposition 4.6.4 (Energy comparison)**.** *Assuming* (BA_∞) *holds with* δ_0*, the three energy-type quantities above are compatible.*

Proof. By Proposition 4.6.3, it suffices to compare the terms

$$\mathcal{Q}[\psi;g](T,(-\mathrm{d}\tau)^{g\sharp}), \qquad \mathcal{Q}[\psi;\mathfrak{m}](T,(-\mathrm{d}\tau)^{\mathfrak{m}\sharp}).$$

We note first that their difference is given by

$$T\psi[(\mathrm{d}\tau)^{g\sharp}-(\mathrm{d}\tau)^{\mathfrak{m}\sharp}]\psi - \frac{1}{2}T(\tau)(g-\mathfrak{m})^{-1}(\mathrm{d}\psi,\mathrm{d}\psi).$$

We can expand this to be schematically

$$\begin{aligned} T\psi\Big[\phi\mathscr{P}_0\frac{u}{\tau}\mathscr{W}_1(L^1\psi+T\psi)+\mathscr{G}\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi)\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\tau)\Big] \\ +\frac{y^0}{\tau}\Big[\phi\mathscr{P}_0\mathscr{W}_2(L^1\psi+T\psi)^2+\mathscr{G}(\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi))^2\Big]. \end{aligned}$$

Hence we can bound the expression by, using (BA_∞) and Proposition 4.6.3,

$$\begin{aligned} \lesssim \frac{\delta_0}{\tau^2 y^0}\mathscr{P}_0|T\psi(L^1\psi+T\psi)| + \frac{\delta_0\tau^\gamma}{(y^0)^{3/2}}|T\psi\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi)| \\ +\frac{\delta_0 y^0}{\tau^2}\mathscr{P}_0\mathscr{W}_2(L^1\psi+T\psi)^2+\frac{y^0}{\tau}(\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi))^2. \end{aligned} \tag{4.6.8}$$

The first term in (4.6.8) can be bounded by

$$\lesssim \frac{\delta_0}{\tau^2}\frac{1}{\cosh(\rho)}|T\psi|^2 + \frac{\delta_0}{\tau^2}\frac{1}{\tau^2\cosh(\rho)}|L^1\psi|^2$$

and the third term by

$$\lesssim \frac{\delta_0}{\tau}\frac{1}{\tau^2\cosh(\rho)}|L^1\psi|^2 + \frac{\delta_0}{\tau^3}\frac{1}{\cosh(\rho)}|T\psi|^2.$$

Both are bounded obviously by a small multiple of $\mathcal{Q}[\psi;\mathfrak{m}](T,(-\mathrm{d}\tau)^{\mathfrak{m}\sharp})$. We can evaluate

$$\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi) = -\frac{\tau^2}{(y^0)^2}T\phi T\psi + \mathscr{B}_1^{1,0}\phi T\psi + \mathscr{B}_1^{1,0}\psi T\phi + \mathscr{B}_1^{1,0}\phi\mathscr{B}_1^{1,0}\psi + \phi\mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)(L^1\psi + T\psi).$$

This implies

$$|\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi)| \lesssim \frac{\delta_0\tau^\gamma}{(y^0)^{3/2}}|T\psi| + \frac{\delta_0\tau^\gamma}{(y^0)^{3/2}\tau}|\mathscr{B}_0^{1,0}\psi|.$$

Thus the second term in (4.6.8) can be bounded by

$$\lesssim \frac{(\delta_0)^2}{\tau^{3-2\gamma}\cosh(\rho)^2}\left(\frac{1}{\cosh(\rho)}|T\psi|^2 + \frac{1}{\tau^2\cosh(\rho)}\sum|L^i\psi|^2\right),$$

and the fourth term by

$$\lesssim \frac{(\delta_0)^2}{\tau^{3-2\gamma}\cosh(\rho)}\left(\frac{1}{\cosh(\rho)}|T\psi|^2 + \frac{1}{\tau^2\cosh(\rho)}\sum|L^i\psi|^2\right).$$

Both terms are similarly controlled by a small multiple of $\mathcal{Q}[\psi;\mathfrak{m}](T,(-\mathrm{d}\tau)^{\mathfrak{m}\sharp})$. This implies our proposition. □

In Section 4.7 below where we treat the inhomogeneous terms, we frequently need to estimate weighted L^2 integrals along Σ_τ. We can compare such integrals to the energies by the following Corollary, which follows after noting $y^0 = \tau\cosh(\rho)$.

Corollary 4.6.5. *We have the following bounds for L^2 integrals of derivatives of ϕ:*

$$\|(y^0\tau)^{-1/2}\mathscr{B}_0^{k,0}\phi\|_{L^2(\Sigma_\tau)} \lesssim \mathfrak{E}_{k-1}(\tau),$$

$$\|(y^0\tau)^{-1/2}\mathscr{B}_0^{k,0}L^1\phi\|_{L^2(\Sigma_\tau)} \lesssim \mathfrak{F}_k(\tau),$$

$$\|(y_0)^{-1/2}\tau^{1/2}\mathscr{B}_1^{k+1,1}\phi\|_{L^2(\Sigma_\tau)} \lesssim \mathfrak{E}_k(\tau),$$

$$\|(y^0)^{-1/2}\tau^{1/2}\mathscr{B}_1^{k+1,1}L^1\phi\|_{L^2(\Sigma_\tau)} \lesssim \mathfrak{F}_{k+1}(\tau).$$

4.6.4 Improved L^∞ bounds from (BA$_2$)

As a consequence of the energy comparison Proposition 4.6.4, we can apply Proposition 4.3.14 with $d = 3$ to (BA$_2$) and derive the following L^∞ estimates of ϕ and its derivatives.

$$\begin{cases} |\phi| + |L^1\phi| \lesssim \dfrac{\delta}{(y^0)^{1/2}}, \\ |\mathscr{B}_0^{1,0}\phi| + |\mathscr{B}_0^{1,0}L^1\phi| \lesssim \dfrac{\delta\tau^\gamma}{(y^0)^{1/2}}, \\ |T\phi| + |TL^1\phi| \lesssim \dfrac{\delta\tau^\gamma}{(y^0)^{1/2}\tau}, \\ |\mathscr{B}_0^{2,0}\phi| \lesssim \dfrac{\delta\tau^{1+\gamma}}{(y^0)^{1/2}}, \\ |\mathscr{B}_1^{2,1}\phi| \lesssim \dfrac{\delta\tau^\gamma}{(y^0)^{1/2}}, \\ |\mathscr{B}_1^{3,1}\phi| \lesssim \dfrac{\delta\tau^{1+\gamma}}{(y^0)^{1/2}}. \end{cases} \tag{4.6.9}$$

With the aid of (A.3.2), we can also estimate

$$\begin{cases} |TT\phi| \lesssim \delta\tau^{\gamma-1}, \\ |\mathscr{B}_0^{1,0}TT\phi| \lesssim \delta\tau^\gamma. \end{cases} \tag{4.6.10}$$

Here we also made use of Lemma 4.6.1 freely.

Remark 4.6.6. Note that we have estimated (4.6.10) by directly estimating the right hand side of (A.3.2) using (4.6.9). In particular these were *not* derived from applying Proposition 4.3.14 to appropriate energy integrals: in fact we have not yet proven any L^2 estimates for $TT\phi$ and its higher derivatives. It turns out the necessary L^2 estimates require a little bit of work, and we defer their proofs to Lemma 4.7.1.

Remark 4.6.7. Notice that (4.6.9) and (4.6.9) controls up to two derivatives of ϕ in all directions, and in particular controls the first derivative of the dynamical metric g. Thus we can apply the blow-up criterion for quasilinear wave equations and assert that the *a priori* estimates guaranteed by our bootstrap argument suffices to prove global existence of the solution.

Remark 4.6.8. In the bootstrap argument we will be studying energies to the top order $\mathfrak{E}_4$ and $\mathfrak{F}_3$, which corresponds to 3 additional derivatives applied to the equations (4.5.12) and (4.5.11) respectively. Examining the terms that show up in the nonlinearities, which depend only on up-to-two derivatives of ϕ, this means that when performing energy estimates the highest derivative that we will put into L^∞ would be three; and as we will only be commuting with $\mathscr{B}_0^{1,0}$ derivatives, there will be no $TTT\phi$ terms to worry about. Hence between (4.6.9) and (4.6.10) all possible L^∞ terms are captured.

4.6.5 Controlling the deformation tensor term

Now let us return to studying the first integrand in (4.6.1) as promised. First, using Proposition 4.6.3, the space-time integral with regards to dvol_g can be replaced by the integral with regards to $\mathrm{dvol}_{\mathfrak{m}}$ to which we can apply the co-area formula and decompose as $\mathrm{dvol}_{\eta_\tau}\ \mathrm{d}\tau$. The same proposition also implies we can replace the hypersurface volume element and have the integral conducted with respect to $\mathrm{dvol}_{h_\tau}\ \mathrm{d}\tau$.

For the integration along Σ_τ, we will put ψ, which is automatically *top order*, in the appropriate weighted L^2 space; by Proposition 4.6.4 these L^2 integrals can be bounded by the quasilinear energies. We therefore obtain the following bound

$$\iint\limits_{\tau\in[\tau_0,\tau_1]} |\mathcal{Q}[\psi;g] :_g \mathcal{L}_T g| \ \mathrm{dvol}_g \le \int_{\tau_0}^{\tau_1} \Big\|\frac{1}{\cosh(\rho)}\mathscr{P}_0(\phi+T\phi)\Big\|_{L^\infty(\Sigma_\tau)} \mathcal{E}_\tau[\psi;g]^2 \\ + \Big\|\cosh(\rho)\mathscr{G}\Big[(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_2^{2,2}\phi) + (\mathscr{B}_1^{1,1}\phi)^2(1+\mathscr{P}_0 T\phi)\Big]\Big\|_{L^\infty(\Sigma_\tau)} \mathcal{E}_\tau[\psi;g]^2 \ \mathrm{d}\tau. \quad (4.6.11)$$

The terms in L^∞ can be estimated with the help of (4.6.9) and (4.6.10). First we have

$$\Big|\frac{1}{\cosh(\rho)}\mathscr{P}_0(\phi+T\phi)\Big| \lesssim \frac{1}{\tau}\Big(\frac{\delta}{\tau} + \frac{\delta\tau^\gamma}{\tau^2}\Big) \le \delta\tau^{-2};$$

we used here that $\frac{1}{\cosh(\rho)} \approx \tau^{-1}$ by Lemma 4.6.1. Next we have

$$|\cosh(\rho)\mathscr{G}\mathscr{B}_1^{1,1}\phi\mathscr{B}_2^{2,2}\phi| \lesssim \frac{y^0}{\tau}\frac{\delta\tau^\gamma}{\sqrt{y^0}\tau}\frac{\delta\tau^\gamma}{\tau} \le \delta^2\tau^{2\gamma-2}$$

after observing Lemma 4.6.1 again. Finally the last term

$$|\cosh(\rho)\mathcal{G}(\mathscr{B}_1^{1,1}\phi)^2(1+\mathscr{P}_0 T\phi)| \lesssim \frac{y^0}{\tau}\frac{\delta^2\tau^{2\gamma}}{y^0\tau^2}\Big(1+\frac{\delta\tau^\gamma}{\sqrt{y^0}\tau}\Big) \lesssim \delta^2\tau^{2\gamma-3}.$$

Hence, with our assumption that $\gamma < 1/4$ we have that

$$\iint\limits_{\tau\in[\tau_0,\tau_1]} |\mathcal{Q}[\psi;g]:_g \mathcal{L}_T g| \,\mathrm{dvol}_g \lesssim \int_{\tau_0}^{\tau_1} \delta\tau^{-3/2}\mathcal{E}_\tau[\psi;g]^2 \,\mathrm{d}\tau. \tag{4.6.12}$$

Note the integrable power in τ: the deformation tensor term does not cause any difficulty in the analysis.

4.7 Controlling the inhomogeneity

In this section we focus our attention on estimating the second term in the energy estimate (4.6.1), given by the integral

$$\iint\limits_{\tau\in[\tau_0,\tau_1]} \Big|\Box_g\psi\cdot T\psi\Big| \mathrm{dvol}_g.$$

By virtue of the geometric comparison Proposition 4.6.3 and the energy comparison Proposition 4.6.4, we can bound this by

$$\int_{\tau_0}^{\tau_1} \Big\|\sqrt{\frac{y^0}{\tau}}\Box_g\psi\Big\|_{L^2(\Sigma_\tau)} \mathcal{E}[\psi;g]\,\mathrm{d}\tau.$$

We will take ψ here one of $\{\phi, L^1\phi, L^\alpha L^1\phi, L^i\phi, L^\alpha L^i\phi\}$, where α is some multi-index with length no more than 3, and $i\in\{2,3\}$.

To streamline our control for the higher derivative terms, we observe the following principle:

$$\Big\|\sqrt{\frac{y^0}{\tau}}(\mathrm{expr})\Big\|_{L^2(\Sigma_\tau)} \lesssim \tau^\nu \implies \Big\|\sqrt{\frac{y^0}{\tau}}\mathscr{B}_0^{1,0}(\mathrm{expr})\Big\|_{L^2(\Sigma_\tau)} \lesssim \tau^{\nu+1}. \tag{SP}$$

Here, (expr) means some polynomial expressions in $\mathcal{G}, \mathscr{P}_*, \mathscr{W}_*$, and $\mathscr{B}_*^{*,1}\phi$. We emphasize that (SP) is a principle meta to our proof, where we will bound each term in the polynomial expression either in some weighted L^2 space on Σ_τ or in L^∞, using the bootstrap assumptions (BA_2) and their consequences (4.6.9) and (4.6.10). The symbol "$\lesssim$" in (SP) should be understood to mean "can be proven as the result of our bootstrap argument to be bounded by", and not a factual assertion of a possibly better bound.

Understood this way, (SP) follows simply from the facts that:

- For $\mathscr{B}_*^{*,1}\phi$ terms, in (BA_2), each higher derivative brings at most an additional loss of τ.
- The terms $\mathscr{W}_*$ are invariant under action by L-derivatives.
- As discussed after Notation 4.5.2, $\mathscr{B}_0^{1,0}\mathscr{P}_m = \mathscr{P}_{m+1}$, which allows it to grow with an additional factor of $\underline{u}^{1/2}$. By Lemma 4.6.1 this can be bounded by τ.
- Finally, observe that

$$\mathscr{B}_0^{1,0}\mathscr{G} = \mathscr{G}\cdot\left[\mathscr{B}_0^{1,0}(\phi\mathscr{P}_0) + \mathscr{B}_0^{2,0}\phi + \mathscr{B}_1^{2,1}\phi\right]$$

  by the chain rule. The first and third terms are in fact decaying by (4.6.9), and the middle term is bounded by $\delta\tau^{1/2+\gamma}$, which, since $\gamma < 1/4$, is less than a full order of τ increase in growth.

Occasionally $\mathscr{B}_*^{*,2}\phi$ terms also occur: these are the terms with two T derivatives. Their L^∞ estimates are already captured in (4.6.10) and they can be seen to also obey the schematic principle (SP) where higher derivatives lose factors of τ.

We complement the estimates with the following L^2 version:

Lemma 4.7.1. *For $0 \le k \le 3$, we have*

$$\|\tau^{5/2}(y^0)^{-3/2}\mathscr{B}_0^{k,0}TT\phi\|_{L^2(\Sigma_\tau)} \lesssim \delta\tau^{\max(k-1,0)+\gamma}.$$

Sketch of proof. The proof of this estimate itself is an application of the principle (SP) and the bootstrap assumptions. Observe first that by (A.3.2) that $TT\phi$ can be expanded as $1/\mathfrak{c}_{TT}$ times a polynomial expression in $\mathscr{G}, \mathscr{P}_*, \mathscr{W}_*$, and $\mathscr{B}_*^{*,1}\phi$ to which (SP) can apply.

For convenience call this polynomial expression $\mathcal{O}$. From Corollary 4.6.5 combined with (4.6.9) we have that

$$\begin{aligned}
\|(\tau y^0)^{1/2}\mathscr{B}_2^{2,1}\phi\|_{L^2} &\lesssim \mathfrak{E}_1 \\
\|\tau^{1/2}(y^0)^{-1/2}\mathscr{B}_1^{1,1}\phi\|_{L^2} &\lesssim \mathfrak{E}_0 \\
|(y^0)\mathscr{P}_0\phi\mathscr{W}_1| &\lesssim \delta\tau^{-1} \\
|(y^0)\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^3| &\lesssim \delta^3\tau^{3\gamma-4} \\
|(y^0)(\mathscr{B}_1^{1,1}\phi)^2| &\lesssim \delta^2\tau^{2\gamma-2} \\
|(y^0)(1+\mathscr{P}_1\phi)\mathscr{W}_1(\mathscr{B}_1^{1,1}\phi)^2| &\lesssim \delta^3\tau^{2\gamma-4}.
\end{aligned}$$

Additionally, we pay attention to the quadratic term

$$\|(\tau y^0)^{1/2}\mathscr{P}_0\mathscr{W}_2(L^1\phi+T\phi)^2\|_{L^2} \lesssim \|\mathscr{P}_0(L^1\phi+T\phi)\|_{L^\infty}\mathfrak{E}_0 \lesssim \delta^2\tau^{-1}.$$

Together with the estimate $\mathfrak{c}_{TT} \approx \tau^2/(y^0)^2$ implies the Lemma when $k=0$. Specifically, we have that

$$\|(\tau y^0)^{1/2}\mathcal{O}\|_{L^2} \lesssim \mathfrak{E}_1 + \mathfrak{E}_0\delta\tau^{-1}.$$

Similar arguments show that

$$\|(\tau y^0)^{1/2}\mathscr{B}_0^{1,0}\mathcal{O}\|_{L^2} \lesssim \mathfrak{E}_2 + \delta\mathfrak{E}_1.$$

For this we crucially need Remark A.3.1 which shows that there is no growth arising from first derivatives of $\mathscr{G}$ terms in (A.3.2). (Note that this step requires explicit argument and *not* an appeal to the principle (SP).) For higher derivatives we can appeal to (SP).

For higher k, one also needs to estimate derivatives of $\mathfrak{c}_{TT}$. We observe the following schematic computation

$$\begin{aligned}
\mathscr{B}_0^{k,0}\mathfrak{c}_{TT} = \tau^2\mathscr{W}_2\Big[1+\mathscr{B}_0^{\le k,0}\Big((\mathscr{B}_1^{1,1}\phi)^2 + \frac{1}{\tau^2}(\mathscr{B}_0^{1,0}\phi)^2 \\
+ \frac{1}{\tau^2}(\mathscr{B}_0^{0,0}\phi)\mathscr{P}_0(1+(\mathscr{B}_1^{1,1}\phi)(L^1\phi+T\phi))\Big)\Big]
\end{aligned}$$

The inner term, operated on by $\mathscr{B}_0^{\leq k,0}$ can be bounded by

$$\delta^2\tau^{2\gamma-2}(y^0)^{-1} + \delta\tau^{-3}(1+\delta^2\tau^{\gamma-4})$$

through (4.6.9). By the schematic principle we have that for $k \leq 2$, $\mathscr{B}_0^{k,0}\mathfrak{c}_{TT}$ is bounded by $\tau^2\mathscr{W}_2$. And this shows the Lemma up to $k \leq 2$.

For $k = 3$, we need to consider the case where all derivatives hits on $\mathfrak{c}_{TT}$, since all other terms follow from the principle (SP). In this case we need to essentially estimate something that is schematically the same as

$$\Big\|\tau^{5/2}(y^0)^{-3/2}TT\phi\cdot\Big[(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{4,1}\phi)+ \\ \tau^{-2}(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_0^{4,0}\phi)+\tau^{-2}\phi\mathscr{P}_0(L^1\phi+\mathscr{B}_1^{1,1}\phi)\mathscr{B}_1^{4,1}\phi\Big]\Big\|_{L^2}.$$

Here we group $\tau^{1/2}(y^0)^{-1/2}$ with the $\mathscr{B}_1^{4,1}\phi$ terms, and $\tau^{-1/2}(y^0)^{-1/2}$ with $\mathscr{B}_0^{4,0}\phi$, to bound in L^2 by $\mathfrak{E}_3$. The remaining parts to be controlled in L^∞ boils down to

$$\tau^2(y^0)^{-1}TT\phi[\mathscr{B}_1^{1,1}\phi+\tau^{-1}(\mathscr{B}_0^{1,0}\phi)+\tau^{-2}\phi\mathscr{P}_0(L^1\phi+\mathscr{B}_1^{1,1}\phi)]$$

which can be bounded by

$$\delta^2\tau^{1+\gamma}(y^0)^{-1}[\tau^{\gamma-1}(y^0)^{-1/2}+\delta\tau^{-5}]\lesssim\delta^2\tau^{2\gamma-3/2}$$

and so we see contributes to a lower order term, and the Lemma holds also for $k = 3$. $\square$

This previous Lemma implies we can also extend (SP) to handle also $\mathscr{B}_*^{*,2}\phi$ terms in the expression.

Remark 4.7.2. Note that (SP) gives the *worst case scenario* bound on the higher derivatives of an expression. As one already sees in the proof of the Lemma above, sometimes this worst case bound is not realized. For example, first derivatives of $\mathscr{G}$ do no lose a whole factor of τ even in the worst case, and as seen in the proof of the Lemma above, sometimes derivatives of $\mathscr{G}$ do not lose decay at all. Similarly, going from ϕ to $\mathscr{B}_0^{1,0}\phi$ in L^∞ only entails a τ^γ loss.

However, overall, the schematic principle (SP) cannot be generally improved. This is due to the possible presence of the $\mathscr{P}_*$ terms. Each time a $\mathscr{B}_0^{1,0}$ derivative hits $\mathscr{P}_*$ we necessarily incur a penalty of one factor of τ. This entirely agrees with our semilinear analysis in Section 4.4 where the highest growth rates always accompanies the terms when ℓ_0 is largest (where most derivatives hit on Υ'').

4.7.1 Higher order nonlinear terms

Proposition 4.7.3. *The following bounds hold:*

$$\|(y^0)^{1/2}\tau^{-1/2}\mathrm{HO}_1\|_{L^2(\Sigma_\tau)} \lesssim \delta^3\tau^{3\gamma-4} \tag{4.7.1}$$

$$\|(y^0)^{1/2}\tau^{-1/2}\mathrm{HO}_i\|_{L^2(\Sigma_\tau)} \lesssim \delta^3\tau^{2\gamma-3}. \tag{4.7.2}$$

Proof. We focus first on $\mathscr{G}(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{1,1}L^1\phi)(\mathscr{B}_1^{2,2}\phi)$, the sole cubic term in HO_1. Outside the $TT\phi$ term this can be bounded by

$$\begin{aligned}\|(y^0)^{-1/2}\tau^{-1/2}\mathscr{G}(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{1,1}L^1\phi)(\mathscr{B}_1^{2,1}\phi)\|_{L^2(\Sigma_\tau)}&\\ \lesssim \|\tau^{-1}(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{1,1}L^1\phi)\|_{L^\infty}\mathcal{E}_1(\tau) \lesssim \delta^3\tau^{2\gamma-4}.&\end{aligned}$$

For the $TT\phi$ term we need to add a factor of $\frac{(y^0)}{\tau^2}$: this is because by Proposition 4.6.3 we have

$$|\mathfrak{c}_{TT}^{-1}\mathscr{B}_2^{2,1}\phi| \lesssim \frac{(y^0)^2}{\tau^2}\frac{1}{y^0}\mathscr{B}_1^{2,1}\phi.$$

This gives the bound by

$$\|\tau^{-3}(y^0)(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{1,1}L^1\phi)\|_{L^\infty}\delta \lesssim \delta^3\tau^{2\gamma-5}$$

where we again used Lemma 4.6.1. (The differing decay rates of the two terms stems from the fact that $\mathscr{B}_0^{1,0}\phi$ has additional decay along Σ_τ compared to $T\phi$, but this decay is not seen when taking L^∞ on Σ_τ.)

The quartic and higher order terms can be treated similarly, the details of which we omit here, the general idea being to put the highest order derivative terms in L^2 and lower ones in L^∞. This shows that the quartic and higher order terms in HO_1 can be bounded by $\lesssim \delta^4\tau^{3\gamma-4}$ uniformly. (We remark here that as all the remaining terms are multiplied by a $\mathscr{P}_*$ weight, for their estimates we can consider $(y^0) \approx \tau^2$. This means that the anisotropy between $\mathscr{B}_2^{2,1}\phi$ terms and $TT\phi$ terms that showed up in the cubic term estimates can be avoided.)

For HO_i, the additional cubic term now is a generic $\mathscr{B}_1^{2,1}\phi$ instead of $\mathscr{B}_1^{1,1}L^1\phi$, which means it decays slower by a factor of τ. The additional quartic terms can all be bounded by $\lesssim \delta^4\tau^{2\gamma-3}$, and our claims follow. □

Remark 4.7.4. To estimate $\mathscr{B}_*^{*,1}\phi$ terms using either the energy (and then by the bootstrap (BA_2)) or using a straight-up L^∞ estimate using the peeling estimates in Proposition 4.3.14, we would need any factors of T derivative to be the outermost one. Luckily, commutation reduces the order of derivatives and leaves the weight unchanged (see Proposition 4.3.12), which has the advantage of guaranteeing that the commutator terms have faster decay (by τ^{-1}).

Remark 4.7.5. The quartic term bounds for HO_1 can be improved from $\delta^4\tau^{3\gamma-4}$ to $\delta^4\tau^{4\gamma-5}$, thereby upgrading the overall bound on HO_1 to $\delta^3\tau^{2\gamma-4}$. This improvement comes from noting that the term $\mathscr{G}\mathscr{P}_0(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_2^{2,2}\phi)$ in the definition of HO_1 is actually

$$\mathscr{G}\mathscr{P}_0(L^1\phi)(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_2^{2,2}\phi).$$

As for our purposes these types of improvements are not essential, and does not effect the closing of the bootstrap, we shall not pursue this and myriad other improvements in the higher order terms.

One should however note that for studying the missing case $d = 2$, the above indicates that careful treatment of all quadratic, cubic, *and* quartic nonlinearities will be likely necessary.

4.7.2 Quadratic terms

Now we consider the quadratic nonlinearities. These terms are a bit more delicate and we will include more details of the arguments.

4.7.2.1 Zeroth order case

Looking at (4.5.13), we need to estimate

$$\|(y^0)^{-3/2}\tau^{-1/2}\mathscr{P}_0(L^1\phi+T\phi)^2\|_{L^2(\Sigma_\tau)}.$$

We observe that

$$\|(y^0)^{-3/2}\tau^{-1/2}\mathscr{P}_0(L^1\phi)^2\|_{L^2(\Sigma_\tau)}\lesssim\|(y^0)^{-1}\mathscr{P}_0L^1\phi\|_{L^\infty}\mathcal{E}_0(\tau)\lesssim\frac{\delta^2}{\tau^3}.$$

Here we used that by (4.6.9), our bootstrap assumptions imply $|L^1\phi|\lesssim\delta(y^0)^{-1/2}$. Additionally recall that $y^0\approx\tau^2$ in the presence of $\mathscr{P}_*$. Next, we have

$$\|(y^0)^{-3/2}\tau^{-1/2}(T\phi)^2\|_{L^2(\Sigma_\tau)}\lesssim\|(y^0)^{-1}\tau^{-1}T\phi\|_{L^\infty}\mathcal{E}_0(\tau)\lesssim\frac{\delta^2\tau^\gamma}{\tau^5}.$$

We note here for this term the $\mathscr{W}_2$ term in the nonlinearity is crucial: without it the denominator would only have τ^{-1} which would not have enabled us to close our estimates.

4.7.2.2 First order, $\psi=L^1\phi$

Let us now consider QN_1 (see (A.3.6)). The terms with $(L^1\phi+T\phi)^2$ and $\phi(L^1\phi+T\phi)$ are controlled exactly as the zeroth order term case, by $\delta^2\tau^{-3}$. For the remaining terms, we see first that

$$\|(y^0)^{-3/2}\tau^{-1/2}\mathscr{P}_0T\phi(L^1L^1\phi+TL^1\phi)\|_{L^2(\Sigma_\tau)}\lesssim\|(y^0)^{-1}T\phi\|_{L^\infty}\mathcal{F}_0(\tau)\lesssim\frac{\delta^2\tau^\gamma}{\tau^4}.$$

Similarly

$$\|(y^0)^{-3/2}\tau^{-1/2}\mathscr{P}_0(\phi+L^1\phi)(L^1L^1\phi+TL^1\phi)\|_{L^2(\Sigma_\tau)}\lesssim\frac{\delta^2}{\tau^3}.$$

The final term involves $TT\phi$, for which we can bound

$$\|(y^0)^{-3/2}\tau^{-1/2}\mathscr{P}_0(\phi+L^1\phi)TT\phi\|_{L^2(\Sigma_\tau)}\lesssim\|\tau^{-3}(\phi+L^1\phi)\|_{L^\infty}\mathcal{E}_1(\tau)\lesssim\frac{\delta^2}{\tau^4}.$$

4.7.2.3 First order, $\psi = L^i\phi$

We next consider QN_i (see (A.3.9)). There is a loss compared to the QN_1 terms, which we expect. First,

$$\begin{aligned}&\|(y^0)^{-1/2}\tau^{-1/2}\mathscr{P}_0(\phi + L^1\phi + T\phi)(\mathscr{B}_1^{1,1}L^1\phi + \mathscr{B}_2^{2,2}\phi + \mathscr{B}_1^{1,1}\phi)\|_{L^2(\Sigma_\tau)}\\ &\qquad\lesssim \|\tau^{-1}\mathscr{P}_0(\phi + L^1\phi + T\phi)\|_{L^\infty}\cdot[\mathfrak{F}_0(\tau) + \mathfrak{E}_1(\tau)] \lesssim \frac{\delta^2}{\tau^2}.\end{aligned}$$

Next

$$\begin{aligned}&\|(y^0)^{-3/2}\tau^{-1/2}(\mathscr{P}_0L^i\phi + \mathscr{P}_1\phi)(L^1L^1\phi + TL^1\phi + TT\phi + L^1\phi + T\phi)\|_{L^2(\Sigma_\tau)}\\ &\qquad\lesssim \|(y^0)^{-1}(\mathscr{P}_0\mathscr{B}_0^{1,0}\phi + \mathscr{P}_1\phi)\|_{L^\infty}\cdot[\mathfrak{F}_0(\tau) + \mathfrak{E}_1(\tau)] \lesssim \frac{\delta^2}{\tau^2}.\end{aligned}$$

Finally,

$$\begin{aligned}&\|(y^0)^{-3/2}\tau^{-1/2}\mathscr{P}_1(\phi + L^1\phi + T\phi)(L^1\phi + T\phi)\|_{L^2(\Sigma_\tau)}\\ &\qquad\lesssim \|(y^0)^{-1}\mathscr{P}_1(\phi + L^1\phi + T\phi)\|_{L^\infty}\cdot\mathfrak{E}_0(\tau) \lesssim \frac{\delta^2}{\tau^2}\end{aligned}$$

4.7.2.4 Higher order cases

By Remark 4.5.7, the higher order derivatives $L^\alpha L^i\phi$ where $i = 2,3$ can be treated using (SP). It suffices to consider the higher derivatives of $L^1\phi$. Observe that the principle (SP) can also be applied to the commutator terms: that control of $\mathscr{B}_0^{1,0}[\mathscr{B}_0^{1,0},\Box_g]\psi$ also gives control of $[\mathscr{B}_0^{1,0},\Box_g]\mathscr{B}_0^{1,0}\psi$, since the terms of the latter is schematically a subset of those terms that appears in the former. Hence it suffices to consider the estimates for $[\mathscr{B}_0^{1,0},\Box_g]L^1\phi$.

We treat each of the six quadratic terms in $[\mathscr{B}_0^{1,0},\Box_g]L^1\phi$ listed in the schematic decomposition (4.5.14) below. First, we can estimate

$$\|(y^0)^{-1/2}\tau^{-1/2}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)(L^1\psi + T\psi)\|_{L^2(\Sigma_\tau)} \lesssim \|\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)\|_{L^\infty}\cdot\mathfrak{F}_0(\tau) \lesssim \frac{\delta^2\tau^\gamma}{\tau^2}.$$

Next, we have

$$\|(y^0)^{-1/2}\tau^{-1/2}\mathscr{P}_0(\phi + L^1\phi + T\phi)(\mathscr{B}_1^{1,1}\psi)\|_{L^2(\Sigma_\tau)} \lesssim \|\tau^{-1}\mathscr{P}_0(\phi + L^1\phi + T\phi)\|_{L^\infty} \cdot \mathsf{F}_0(\tau) \lesssim \frac{\delta^2}{\tau^2}.$$

The third term we estimate by

$$\begin{aligned}&\|(y^0)^{-1/2}\tau^{-1/2}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{2,1}\psi + TT\psi)\|_{L^2(\Sigma_\tau)} \\ &\qquad \lesssim \|\tau^{-1}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)\|_{L^\infty} \cdot [\mathsf{F}_1(\tau) + \mathsf{E}_2(\tau)] \lesssim \frac{\delta^2\tau^{2\gamma}}{\tau^3}.\end{aligned}$$

Next we have

$$\begin{aligned}&\|(y^0)^{-3/2}\tau^{-1/2}\mathscr{P}_0(\mathscr{B}_0^{1,0}L^1\phi + \mathscr{B}_0^{1,0}T\phi)(L^1\psi + T\psi)\|_{L^2(\Sigma_\tau)} \\ &\qquad \lesssim \|(y^0)^{-1}\mathscr{P}_0(\mathscr{B}_0^{1,0}L^1\phi + \mathscr{B}_0^{1,0}T\phi)\|_{L^\infty} \cdot \mathsf{F}_1(\tau) \lesssim \frac{\delta^2\tau^{\gamma}}{\tau^3}.\end{aligned}$$

The fifth term we estimate by

$$\begin{aligned}&\|(y^0)^{-3/2}\tau^{-1/2}\mathscr{P}_1(\phi + L^1\phi + T\phi)(L^1\psi + T\psi)\|_{L^2(\Sigma_\tau)} \\ &\qquad \lesssim \|(y^0)^{-1}\mathscr{P}_1(\phi + L^1\phi + T\phi)\|_{L^\infty} \cdot \mathsf{F}_1(\tau) \lesssim \frac{\delta^2}{\tau^2}.\end{aligned}$$

And the final term is estimated by

$$\begin{aligned}&\|(y^0)^{-3/2}\tau^{-1/2}\mathscr{P}_1\phi(L^1L^1\psi + TL^1\psi + TT\psi)\|_{L^2(\Sigma_\tau)} \\ &\qquad \lesssim \|(y^0)^{-1}\mathscr{P}_1\phi\|_{L^\infty} \cdot [\mathsf{F}_1(\tau) + \mathsf{E}_2(\tau)] \lesssim \frac{\delta^2\tau^{\gamma}}{\tau^2}.\end{aligned}$$

4.8 Closing the bootstrap

We conclude our proof of Theorem 4.5.8 by putting together the estimates in the previous sections using (4.6.1). By our control of the deformation tensor (4.6.12), we have that

$$\mathcal{E}_{\tau_1}[\psi;g]^2 - \mathcal{E}_{\tau_0}[\psi;g] \lesssim \int_{\tau_0}^{\tau_1} \frac{\delta}{\tau^{3/2}}\mathcal{E}_\tau[\psi;g]^2 + \|(y^0)^{1/2}\tau^{-1/2}\Box_g\psi\|_{L^2(\Sigma_\tau)}\mathcal{E}_\tau[\psi;g]\,\mathrm{d}\tau.$$

Now let $\sigma \in (2, T)$.

From our bootstrap assumptions (BA_2) and the computations of Section 4.7.2.1, we get

$$\mathfrak{E}_0(\sigma)^2 - \mathfrak{E}_0(2)^2 \lesssim \int_2^\sigma \frac{\delta^3}{\tau^{3/2}} + \frac{\delta^3}{\tau^3} \,\mathrm{d}\tau \lesssim \delta^3. \tag{4.8.1}$$

From Section 4.7.2.2, we get

$$\mathfrak{F}_0(\sigma)^2 - \mathfrak{F}_0(2)^2 \lesssim \int_2^\sigma \frac{\delta^3}{\tau^{3/2}} + \frac{\delta^3}{\tau^3} \,\mathrm{d}\tau \lesssim \delta^3. \tag{4.8.2}$$

From Section 4.7.2.3, we get

$$\mathfrak{E}_1(\sigma)^2 - \mathfrak{E}_1(2)^2 \lesssim \int_2^\sigma \frac{\delta^3}{\tau^{3/2}} + \frac{\delta^3}{\tau^2} \,\mathrm{d}\tau \lesssim \delta^3. \tag{4.8.3}$$

By applying the principle (SP) and factoring in Remark 4.5.7 this implies for $k \geq 2$,

$$\begin{aligned}\mathfrak{E}_k(\sigma)^2 - \mathfrak{E}_k(2)^2 &\lesssim \int_2^\sigma \frac{\delta^3 \tau^{2\gamma+2(k-2)}}{\tau^{3/2}} + \frac{\delta^3 \tau^{\gamma+k-2}}{\tau^2} \cdot \tau^{k-1} \,\mathrm{d}\tau \\ &\lesssim \delta^3 \sigma^{2\gamma+2(k-2)-1/2} + \delta^3 \sigma^{\gamma+2(k-2)} \leq \delta^3 \sigma^{2\gamma+2(k-2)}. \end{aligned} \tag{4.8.4}$$

Finally, from Section 4.7.2.4 and principle (SP) we get

$$\mathfrak{F}_1(\sigma)^2 - \mathfrak{F}_1(2)^2 \lesssim \int_2^\sigma \frac{\delta^3}{\tau^{3/2}} + \frac{\delta^3}{\tau^2} + \frac{\delta^3 \tau^\gamma}{\tau^2} \,\mathrm{d}\tau \lesssim \delta^3. \tag{4.8.5}$$

Further applications of the principle (SP) gives us the higher order estimates for $k \geq 2$

$$\begin{aligned}\mathfrak{F}_k(\sigma)^2 - \mathfrak{F}_k(2)^2 &\lesssim \int_2^\sigma \frac{\delta^3 \tau^{2\gamma+2(k-2)}}{\tau^{3/2}} + \frac{\delta^3 \tau^{\gamma+k-2}}{\tau^2} \cdot \tau^{k-1} + \frac{\delta^3 \tau^{2\gamma+k-2}}{\tau^2} \cdot \tau^{k-1} \,\mathrm{d}\tau \\ &\lesssim \delta^3 \sigma^{2\gamma+2(k-2)-1/2} + \delta^3 \sigma^{\gamma+2(k-2)} + \delta^3 \sigma^{2\gamma+2(k-2)} \leq \delta^3 \sigma^{2\gamma+2(k-2)}. \end{aligned} \tag{4.8.6}$$

With these estimates, the bootstrap closes provided δ, ϵ are taken sufficiently small.

We close our discussion with a couple of remarks.

Remark 4.8.1. One interesting aspect of our argument is that the semilinear nonlinearities seem to allow closing the bootstrap using only a $\log(\tau)$ loss instead of τ^γ. This is seemingly in contradiction to the discussion in Section 4.4, where τ^γ losses seems to be necessary when the dimension $d = 3, 4$. The explanation for this is that in our semilinear analyses we did *not* separate out the privileged direction $L^1\phi$ as having better decay properties. Had we also isolated the direction $L^1\phi$ and run the argument with separate energies for generic derivatives and derivatives with at least one L^1 vector field, we would find also that it is possible to close the argument with merely a $\log(\tau)$ loss at energy level $d-1$, with a further τ loss with each additional derivative, analogously to the cases where $d \geq 5$.

As discussed at the start of Section 4.5, one would see additional losses for the full quasilinear problem were one not to separate out the better direction L^1. This is reflected in the fact that the part where we *required* the τ^γ loss in place of a mere log-loss occurs in Section 4.7.2.4, where we considered the effects of the commutator term $[X, \Box_g]\psi$; note of course that the commutator term vanishes for our semilinear model problem.

Remark 4.8.2. One may ask whether the higher energy growth is associated to the *blow-up at infinity* described by Alinhac [Ali03], and which seems generic for wave equations with weak-null quasilinearities [Lin08, LR05, DP18]. This seems *not* to be the case for several reasons. First among the reasons is that we observe the same higher energy growth even for the semilinear model considered in Section 4.4. Additionally, our energy growth is not very severe; when translated back to L^∞ estimates of the coordinate derivatives, we still observe decay (though at a reduced rate compared to what is available for the linear wave equation). Finally, examining the leading order correction of the quasilinear metric is given with the coefficients $\phi\Upsilon''\mathrm{d}u \otimes \mathrm{d}u$. The localization by the Υ'' term means that the slowly decaying coefficients are supported away from future time-like infinity. This appears in contrast to the known manifestations of blow-up at infinity where the null structure of the dynamical metric is significantly different from the Minkowskian one near future time-like infinity.

APPENDIX

A.1 Tools from analysis

The following are the standard Sobolev inequalities on $\mathbb{R}^d$ that we use freely throughout this work.

Theorem A.1.1 (Sobolev inequalities)**.** *For $u \in C_c^\infty(\mathbb{R}^d)$, for any $1 \le p < d$, there exists a constant C depending only on p and d such that*

$$\|u\|_{L^{d/(d-p)}(\mathbb{R}^d)} \le C\|\nabla u\|_{L^p(\mathbb{R}^d)}. \tag{A.1.1}$$

For $u \in C_c^\infty(\mathbb{R}^d)$, for $k \in \mathbb{N}$ and $d < kp$, there exists a constant depending only on p, k and d such that

$$\|u\|_{L^\infty(\mathbb{R}^d)} \le C \sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^p(\mathbb{R}^d)}. \tag{A.1.2}$$

Remark A.1.2. In the special case that $p = 1$, (A.1.2) holds at the end point $k = d$.

The following is an abstract formulation of the bootstrap principle, adapted from [Tao06, Proposition 1.21].

Proposition A.1.3 (Abstract bootstrap principle)**.** *Let $I \subset \mathbb{R}$ be a time interval and for each $t \in I$ let $\mathbf{C}(t)$ be a boolean valued function. Suppose one can verify the following statements:*

1. (Conclusion holds somewhere). *There exists a $t_0 \in I$ such that $\mathbf{C}(t_0)$ is true.*

2. (Conclusion is continuous) *If $\{t_n\}_{n=1}^\infty \subset I$ is a sequence converging to t_∞ and $\mathbf{C}(t_n)$ is true for all n, then $\mathbf{C}(t_\infty)$ is true.*

3. (Conclusion implies better than inclusion.) *If $\mathbf{C}(t_0) = 1$ for some $t_0 \in I$, then there exists some $\epsilon > 0$ such that $(t_0 - \epsilon, t_0 + \epsilon) \subset I$ and $\mathbf{C}(t)$ is true on $(t_0 - \epsilon, t_0 + \epsilon)$.*

Then $\mathbf{C}(t)$ is true on all of I.

Proof. The proposition is an immediate consequence of the continuity principle: *Let X be a connected topological space with a non-empty, open, and closed subspace $Y \subset X$. Then $Y = X$.* □

A.2 Hyperboloidal polar coordinates and related estimates

In this appendix we prove several estimates found in Chapter 2 for which a proof was not provided. The proofs provided are adapted from [Won17b] and are not original material. The proofs are technicalIy involved, and since they do not contribute to the points made in Chapter 2, but for the sake of completeness, we decided to provide them here.

A.2.1 Geometric setup

We will be doing our computations on a hyperbolic coordinate system. Identify $\mathbb{S}^{d-1}$ canonically as a submanifold of $\mathbb{R}^d$ with coordinates θ. Then let

$$(\tau, \rho, \theta) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{S}^{d-1}$$

be the coordinates defined by

$$t = \tau \cosh(\rho); \tag{A.2.1}$$

$$x = \tau \sinh(\rho)\theta. \tag{A.2.2}$$

In this coordinate system the Minkowski metric takes the form

$$\mathfrak{m} = -\mathrm{d}\tau^2 + \tau^2 \mathrm{d}\rho^2 + \tau^2 \sinh^2(\rho)\mathrm{d}\theta^2,$$

where $\mathrm{d}\theta^2$ is the Euclidean metric on $\mathbb{S}^{d-1}$. The induced Riemannian metric and its inverse on Σ_τ are

$$h_\tau = \tau^2(\mathrm{d}\rho^2 + \sinh^2(\rho)\mathrm{d}\theta^2); \tag{A.2.3}$$

$$(h_\tau)^{-1} = \frac{1}{\tau^2}\left(\partial_\rho \otimes \partial_\rho + \frac{1}{\sinh^2(\rho)}\partial_\theta \otimes \partial_\theta\right). \tag{A.2.4}$$

Here $\partial_\theta \otimes \partial_\theta$ is the Euclidean inverse metric on $\mathbb{S}^{d-1}$. We claim that

$$\sum_{i=1}^{d} L^i \otimes L^i = \partial_\rho \otimes \partial_\rho + \frac{t^2}{r^2}\partial_\theta \otimes \partial_\theta. \tag{A.2.5}$$

We compute with the chain rule

$$\begin{aligned}\partial_\rho &= \frac{\partial t}{\partial \rho}\partial_t + \frac{\partial x}{\partial \rho}\cdot \partial_x \\ &= \tau \sinh(\rho)\partial_t + \tau\cosh(\rho)\theta \cdot \partial_x. \end{aligned} \tag{A.2.6}$$

Then

$$\begin{aligned}\partial_\rho \otimes \partial_\rho &= \tau^2 \sinh^2(\rho)\partial_t \otimes \partial_t + \tau\cosh(\rho)(\partial_t \otimes \tau \sinh(\rho)\theta\cdot\partial_x + \tau\sinh(\rho)\theta\cdot\partial_x \otimes \partial_t) \\ &\quad + \tau^2\cosh^2(\rho)\theta\cdot\partial_x \otimes \theta\cdot\partial_x \\ &= |x|^2\partial_t\otimes\partial_t + t(\partial_t \otimes x\cdot\partial_x + x\cdot\partial_x\otimes\partial_t) + t^2\theta\cdot\partial_x\otimes\theta\cdot\partial_x.\end{aligned}$$

Since $x = r\theta$ and $x\cdot\partial_x = r\partial_r$,

$$\partial_\rho\otimes\partial_\rho = r^2\partial_t\otimes\partial_t + tr(\partial_t\otimes\partial_r + \partial_r\otimes\partial_t) + t^2\partial_r\otimes\partial_r.$$

We compute from $\theta = \frac{x}{r}$

$$\begin{aligned}\mathrm{d}\theta &= \frac{1}{r}\mathrm{d}x - \frac{1}{r^2}x\mathrm{d}r; \\ \mathrm{d}\theta\otimes\mathrm{d}\theta &= \frac{1}{r^2}\mathrm{d}x\otimes\mathrm{d}x - \frac{x}{r^3}(\mathrm{d}x\otimes\mathrm{d}r + \mathrm{d}r\otimes\mathrm{d}x) + \frac{|x|^2}{r^4}\mathrm{d}r\otimes\mathrm{d}r \\ &= \frac{1}{r^2}\mathrm{d}x\otimes\mathrm{d}x - \frac{1}{r^2}\mathrm{d}r\otimes\mathrm{d}r; \\ \partial_\theta\otimes\partial_\theta &= r^2\partial_x\otimes\partial_x - r^2\partial_r\otimes\partial_r.\end{aligned}$$

Then we finally compute

$$\begin{aligned}\sum_{i=1}^d L^i\otimes L^i &= \sum_{i=1}^d t^2\partial_i\otimes\partial_i + tx^i(\partial_i\otimes\partial_t + \partial_t\otimes\partial_i) + (x^i)^2\partial_t\otimes\partial_t \\ &= t^2\partial_x\otimes\partial_x + tr(\partial_r\otimes\partial_t + \partial_t\otimes\partial_r) + r^2\partial_t\otimes\partial_t \\ &= t^2\partial_x\otimes\partial_x - t^2\partial_r\otimes\partial_r + tr(\partial_r\otimes\partial_t + \partial_t\otimes\partial_r) + r^2\partial_t\otimes\partial_t + t^2\partial_r\partial_r \\ &= \frac{t^2}{r^2}\partial_\theta\otimes\partial_\theta + \partial_\rho\otimes\partial_\rho.\end{aligned}$$

This finally concludes the proof of (A.2.5). Of course, this also implies

$$\sum_{i=1}^d L^i\otimes L^i = \partial_\rho\otimes\partial_\rho + \frac{\cosh^2(\rho)}{\sinh^2(\rho)}\partial_\theta\otimes\partial_\theta. \tag{A.2.7}$$

On the other hand we have

$$\begin{aligned}\sum_{i<j}\Omega_{ij}\otimes\Omega_{ij} &= \sum_{i<j}(x^i\partial_{x^j} - x^j\partial_{x^i})\otimes(x^i\partial_{x^j} - x^j\partial_{x^i})\\ &= \sum_{i<j}(x^i)^2\partial_{x^j}\otimes\partial_{x^j} + (x^j)^2\partial_{x^i}\otimes\partial_{x^i} - x^ix^j(\partial_{x^j}\otimes\partial_{x^i} + \partial_{x^i}\otimes\partial_{x^j})\\ &= r^2\sum_i \partial_{x^i}\otimes\partial_{x^i} - r^2\partial_r\otimes\partial_r\\ &= \partial_\theta\otimes\partial_\theta \end{aligned} \tag{A.2.8}$$

These computations help us conclude

$$\begin{aligned}(\tau^{-2}h_\tau)^{-1} + \sum_{i<j}\Omega_{ij}\otimes\Omega_{ij} &= \partial_\rho\otimes\partial_\rho + \frac{1}{\sinh^2(\rho)}\partial_\theta\otimes\partial_\theta + \partial_\theta\otimes\partial_\theta\\ &= \partial_\rho\otimes\partial_\rho + \frac{1+\sinh^2(\rho)}{\sinh^2(\rho)}\partial_\theta\otimes\partial_\theta\\ &= \partial_\rho\otimes\partial_\rho + \frac{\cosh^2(\rho)}{\sinh^2(\rho)}\partial_\theta\otimes\partial_\theta\\ &= \sum_{i=1}^d L^i\otimes L^i. \end{aligned} \tag{A.2.9}$$

Note that $\tau^{-2}h_\tau$ is the standard metric on hyperbolic space $\mathbb{H}^d$. Let ∇ be the Levi-Civita connection on Σ_τ relative to the metric h_τ. Contracting (A.2.9) with $df = \nabla f$ shows

$$\langle\nabla f, \nabla f\rangle_{\tau^{-2}h_\tau} \leq \sum_{i=1}^d |L^i f|^2. \tag{A.2.10}$$

A.2.2 Proof of global Sobolev pointwise estimate

In this subsection we prove Theorem 2.1.1. The Sobolev inequality on the unit disk implies the following version in a bounded region of hyperbolic space.

Proposition A.2.1. *Let $f : \mathbb{H}^d \to \mathbb{R}$ be a function on hyperbolic space, which we represent in* *polar coordinates* $(\rho,\theta) \in \mathbb{R}_+ \times S^{d-1}$. *Then*

$$\sup_{\rho<\frac{5}{3}} |f(\rho,\theta)|^2 \lesssim \sum_{k \le \lfloor \frac{d}{2} \rfloor + 1} \int_0^2 \int_{\mathbb{S}^{d-1}} |\nabla^k f|_h^2 \sinh^{d-1}(\rho)\, \mathrm{d}\theta \mathrm{d}\rho. \tag{A.2.11}$$

Remark A.2.2. In estimate (A.2.11), h corresponds to $\tau^{-2} h_\tau$ on the τ-slices. For convenience we denote dvol_{h_τ} as the volume form induced by h_τ (see (A.2.3)) and $\mathrm{dvol}_{\tau^{-1} h_\tau}$ as the volume form induced by the rescaled metric $\tau^{-2} h_\tau$.

Corollary A.2.3. *Let* $\ell \in \mathbb{R}$ *and* f *be a function on* $\Sigma_\tau \subset \mathbb{R}^{1,d}$. *Then*

$$\sup_{\rho<\frac{5}{3}} |f(\tau,\rho,\theta)|^2 \cosh^{\ell+d-1}(\rho) \lesssim \tau^{-d} \sum_{|\alpha| \le \lfloor \frac{d}{2} \rfloor + 1} \int_{\Sigma_\tau} \cosh^{\ell}(\rho) |L^\alpha f|^2 \, \mathrm{dvol}_{h_\tau}.$$

Proof. Applying (A.2.11) and then (A.2.10) we find

$$\begin{aligned}
\sup_{\rho<\frac{5}{3}} |f(\tau,\rho,\theta)|^2 &\lesssim \sum_{k \le \lfloor \frac{d}{2} \rfloor + 1} \int_{\Sigma_\tau \cap \{\rho<2\}} |\nabla^k f|^2_{\tau^{-2} h_\tau} \, \mathrm{dvol}_{\tau^{-2} h_\tau} \\
&\lesssim \sum_{k \le \lfloor \frac{d}{2} \rfloor + 1} \sum_{|\alpha| \le k} \int_{\Sigma_\tau \cap \{\rho<2\}} |L^\alpha f|^2 \, \mathrm{dvol}_{\tau^{-2} h_\tau}.
\end{aligned}$$

Here we have implicitly used that when $\rho < 2$, the quantities in the lower-order terms $|\nabla_{L^i} L^j|_{\tau^{-2} h_\tau}$ have universal bounds, and the L^i are close to orthogonal. Finally, we note that $\cosh(\rho)$ is universally bounded from above and below on $\rho < 2$ so that we may insert it to our estimates by paying the price of universal constants.

We finally note that

$$\begin{aligned}
\mathrm{dvol}_{h_\tau} &= \sqrt{\det h_\tau}\, \mathrm{d}\rho \wedge \mathrm{d}\theta \\
&= \sqrt{\tau^{2d} \sinh^{2d-2}(\rho)}\, \mathrm{d}\rho \wedge \mathrm{d}\theta \\
&= \tau^d \sinh^{d-1} \, \mathrm{d}\rho \wedge \mathrm{d}\theta \\
&= \tau^d \, \mathrm{dvol}_{\tau^{-2} h_\tau}.
\end{aligned} \tag{A.2.12}$$

From this we conclude

$$\begin{aligned}\sup_{\rho<\frac{5}{3}}|f(\tau,\rho,\theta)|^2\cosh^{\ell+d-1}(\rho) &\lesssim \sum_{|\alpha|\le\lfloor\frac{d}{2}\rfloor+1}\int_{\Sigma_\tau\cap\{\rho<2\}}\cosh^{\ell}(\rho)|L^\alpha f|^2\ \mathrm{dvol}_{\tau^{-2}h_\tau}\\ &\lesssim \tau^{-d}\sum_{|\alpha|\le\lfloor\frac{d}{2}\rfloor+1}\int_{\Sigma_\tau\cap\{\rho<2\}}\cosh^{\ell}(\rho)|L^\alpha f|^2\ \mathrm{dvol}_{h_\tau}\\ &\lesssim \tau^{-d}\sum_{|\alpha|\le\lfloor\frac{d}{2}\rfloor+1}\int_{\Sigma_\tau}\cosh^{\ell}(\rho)|L^\alpha f|^2\ \mathrm{dvol}_{h_\tau},\end{aligned}$$

as desired. □

On $\rho>1$ we can use a different localized Sobolev estimate. The Sobolev inequality on the half-infinite cylinder states that

$$\sup_{\rho>\frac{4}{3}}|f(\rho,\theta)|^2 \lesssim \sum_{|\alpha|\le\lfloor\frac{d}{2}\rfloor+1}\int_1^\infty\int_{\mathbb{S}^{d-1}}|\partial^\alpha f|^2\ \mathrm{d}\theta\mathrm{d}\rho. \qquad \text{(A.2.13)}$$

Proposition A.2.4. *Let $\ell\in\mathbb{R}$ and f be a function on $\Sigma_\tau\subset\mathbb{R}^{1,d}$. Then*

$$\sup_{\rho>\frac{4}{3}}|f(\tau,\rho,\theta)|^2\cosh^{\ell+d-1}(\rho)\lesssim\tau^{-d}\sum_{|\alpha|\le\lfloor\frac{d}{2}\rfloor+1}\int_{\Sigma_\tau}\cosh^{\ell}(\rho)|L^\alpha f|^2\ \mathrm{dvol}_{h_\tau}.$$

Proof. We apply (A.2.13) to the function $f\cosh^{\frac{\ell}{2}}(\rho)\sinh^{\frac{d-1}{2}}(\rho)$. Also note that, since $\rho>1$, the hyperbolic functions $\cosh(\rho)$, $\sinh(\rho)$ are uniformly comparable to e^ρ. Consequently, the terms that arise from the product rule can be bounded as

$$\begin{aligned}|\partial^\alpha(f\cosh^{\frac{\ell}{2}}(\rho)\sinh^{\frac{d-1}{2}}(\rho))|^2 &\lesssim \sum_{|\beta_1|+|\beta_2|+|\beta_3|=|\alpha|}|\partial^{\beta_1}f\partial^{\beta_2}(\cosh^{\frac{\ell}{2}}(\rho))\partial^{\beta_3}(\sinh^{\frac{d-1}{2}}(\rho))|^2\\ &\lesssim\sum_{|\beta|\le|\alpha|}|\partial^\beta f|^2\cosh^{\ell}(\rho)\sinh^{d-1}(\rho).\end{aligned}$$

Equation (A.2.6) shows that ∂_ρ can be written as a linear combination of L^i:

$$\begin{aligned}\partial_\rho &= x \cdot \theta \partial_t + t\theta \cdot \partial_x \\ &= \frac{x}{r} \cdot x\partial_t + \frac{x}{r} \cdot t\partial_x \\ &= \frac{x}{r} \cdot \sum_{i=1}^{d} L^i.\end{aligned}$$

Contracting (A.2.8) with $\mathrm{d}\theta$ implies

$$\partial_\theta = \sum_{i<j} \Omega_{ij}(\theta)\Omega_{ij}.$$

Then

$$1 \geq \Omega_{ij}(\theta_k) = \begin{cases} 0 & k \neq i, j; \\ \frac{x^j}{r} & k = i \vee k = j, \end{cases}$$

and

$$\Omega_{ij} = \frac{x^i}{t} L^j - \frac{x^j}{t} L^i$$

show that ∂_θ can be bounded by a linear combination of $\frac{x}{r} L^i$ (recalling that $t \geq r$). This, and since x/r have bounded derivatives along Σ_τ to all orders away from $\rho = 0$, we conclude that

$$|\partial^\beta f|^2 \lesssim \sum_{|\alpha| \leq |\beta|} |L^\alpha f|^2.$$

We finally conclude with (A.2.13, A.2.12)

$$
\begin{aligned}
\sup_{\rho>\frac{4}{3}} |f(\tau,\rho,\theta)|^2 \cosh^{\ell+d-1}(\rho) &\lesssim \sup_{\rho>\frac{4}{3}} |f(\tau,\rho,\theta)|^2 \cosh^{\ell}(\rho)\sinh^{d-1}(\rho) \\
&\lesssim \sum_{|\alpha|\leq\lfloor\frac{d}{2}\rfloor+1} \int_1^\infty \int_{\mathbb{S}^{d-1}} |\partial^\alpha f|^2 \cosh^\ell(\rho) \underbrace{\sinh^{d-1}(\rho)\, \mathrm{d}\theta \mathrm{d}\rho}_{\mathrm{dvol}_{\tau^{-2}h_\tau}} \\
&\lesssim \sum_{|\alpha|\leq\lfloor\frac{d}{2}\rfloor+1} \int_{\Sigma_\tau\cap\{\rho>1\}} |\partial^\alpha f|^2 \cosh^\ell(\rho)\ \mathrm{dvol}_{\tau^{-2}h_\tau} \\
&\lesssim \sum_{|\alpha|\leq\lfloor\frac{d}{2}\rfloor+1} \int_{\Sigma_\tau\cap\{\rho>1\}} |L^\alpha f|^2 \cosh^\ell(\rho)\ \mathrm{dvol}_{\tau^{-2}h_\tau} \\
&\lesssim \sum_{|\alpha|\leq\lfloor\frac{d}{2}\rfloor+1} \int_{\Sigma_\tau} |L^\alpha f|^2 \cosh^\ell(\rho)\ \mathrm{dvol}_{\tau^{-2}h_\tau} \\
&\lesssim \tau^{-d} \sum_{|\alpha|\leq\lfloor\frac{d}{2}\rfloor+1} \int_{\Sigma_\tau} \cosh^\ell(\rho) |L^\alpha f|^2\ \mathrm{dvol}_{h_\tau},
\end{aligned}
$$

as desired.

□

We are finally ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. Combining Corollary A.2.3 and Proposition A.2.4, we conclude

$$
|f(\tau,\rho,\theta)|^2 \lesssim \tau^{-d} \cosh^{1-\ell-d}(\rho) \sum_{|\alpha|\leq\lfloor\frac{d}{2}\rfloor+1} \int_{\Sigma_\tau} \cosh^\ell(\rho) |L^\alpha f|^2\ \mathrm{dvol}_{h_\tau}
$$

The result follows from the observation that $w_\tau = \tau\cosh(\rho)$ and that τ is constant over Σ_τ. □

Theorem 2.1.1 becomes useful when used in conjunction with Lemma 2.2.5 and Proposition 2.2.2. In order to prove Lemma 2.2.5, we provide an intermediate result which is equivalent to Hardy's inequality adapted to the hyperbolas Σ_τ. This proof is adapted from [Won17b].

Lemma A.2.5. ([Won17b, Lemma 5.1]). *Let $d \geq 3$. For any function u defined on Σ_τ,*

$$\|u\|_{\mathcal{L}^2_{-1}} \leq \frac{2}{d-2}\|u\|_{\mathring{\mathcal{W}}^{1,2}_{-1}}. \tag{A.2.14}$$

Proof. First we claim that if $f : [0,\infty) \to \mathbb{R}$ has compact support, though not necessarily vanishing at zero, then

$$\int_0^\infty f(\rho)^2 \cosh(\rho)\sinh^{\alpha-1}(\rho)\,\mathrm{d}\rho \leq \frac{4}{\alpha^2}\int_0^\infty f'(\rho)^2 \frac{\sinh^{\alpha+1}(\rho)}{\cosh(\rho)}\,\mathrm{d}\rho. \tag{A.2.15}$$

Through the compact support assumption we see

$$0 = \int_0^\infty \partial_\rho\left(f(\rho)^2\sinh^\alpha(\rho)\right)\mathrm{d}\rho = \alpha\int_0^\infty f(\rho)^2\sinh^{\alpha-1}(\rho)\cosh(\rho)\,\mathrm{d}\rho + 2\int_0^\infty f(\rho)f'(\rho)\sinh^\alpha(\rho)\,\mathrm{d}\rho.$$

Then we estimate with Hölder's inequality

$$\begin{aligned}
\alpha\int_0^\infty f^2\sinh^{\alpha-1}(\rho)\cosh(\rho)\,\mathrm{d}\rho &\leq 2\left|\int_0^\infty f(\rho)f'(\rho)\sinh^\alpha(\rho)\,\mathrm{d}\rho\right| \\
&\leq 2\int_0^\infty \left|f(\rho)\cosh^{\frac{1}{2}}(\rho)\sinh^{\frac{\alpha-1}{2}}\right|\left|f'(\rho)\frac{\sinh^{\frac{\alpha+1}{2}}(\rho)}{\cosh^{\frac{1}{2}}(\rho)}\right|\mathrm{d}\rho \\
&\leq 2\left(\int_0^\infty f^2\cosh(\rho)\sinh^{\alpha-1}(\rho)\,\mathrm{d}\rho\right)^{\frac{1}{2}}\left(\int_0^\infty f'^2\frac{\sinh^{\alpha+1}(\rho)}{\cosh(\rho)}\,\mathrm{d}\rho\right)^{\frac{1}{2}}.
\end{aligned}$$

We have thus proved (A.2.15). We now prove . We choose $\alpha = d-2$ so that $\alpha - 1 = d-3$ and $\alpha+1 = d-1$. Then we use the trivial inequality $\sinh(\rho) \leq \cosh(\rho)$ and (A.2.15) to see

$$\begin{aligned}
\int_{\Sigma_\tau}\frac{1}{\cosh(\rho)}\phi^2\,\mathrm{dvol}_{h_\tau} &= \tau^{-d}\int_0^\infty\int_{\mathbb{S}^{d-1}}\frac{1}{\cosh(\rho)}\phi^2\sinh^{d-1}(\rho)\,\mathrm{d}\theta\mathrm{d}\rho \\
&\leq \tau^{-d}\int_0^\infty\int_{\mathbb{S}^{d-1}}\frac{\cosh^2(\rho)}{\cosh(\rho)}\phi^2\sinh^{d-3}(\rho)\,\mathrm{d}\theta\mathrm{d}\rho \\
&\leq \frac{4\tau^{-d}}{(d-2)^2}\int_0^\infty\int_{\mathbb{S}^{d-1}}(\partial_\rho\phi)^2\frac{\sinh^{\alpha+1}(\rho)}{\cosh(\rho)}\,\mathrm{d}\theta\mathrm{d}\rho \\
&= \frac{4}{(d-2)^2}\int_{\Sigma_\tau}\frac{1}{\cosh(\rho)}(\partial_\rho\phi)^2\,\mathrm{dvol}_{h_\tau}.
\end{aligned}$$

The decomposition (A.2.7) concludes the proof.

□

We now prove Lemma 2.2.5.

Proof of Lemma 2.2.5. We begin with the proof of (2.2.12). The chain rule dictates

$$\partial_t = \frac{\partial \tau}{\partial t}\partial_\tau + \frac{\partial \rho}{\partial t}\partial_\rho + \frac{\partial \theta}{\partial t}\cdot\partial_\theta.$$

From $\theta = x/r$ we see $\frac{\partial\theta}{\partial t} = 0$. From $t^2 - |x|^2 = \tau^2$ we find

$$2\tau\cosh(\rho) = 2t = 2\frac{\partial\tau}{\partial t}\tau,$$

so $\frac{\partial\tau}{\partial t} = \cosh(\rho)$. Solving $\rho = \coth^{-1}(\frac{t}{x\cdot\theta})$ we compute

$$\frac{\partial\rho}{\partial t} = \frac{\frac{1}{x\cdot\theta}}{1-\frac{t^2}{|x|^2}} = \frac{\frac{1}{\tau\sinh(\rho)}}{1-\frac{\cosh^2(\rho)}{\sinh^2(\rho)}} = \tau^{-1}\frac{\sinh(\rho)}{\sinh^2(\rho)-\cosh^2(\rho)} = -\tau^{-1}\sinh(\rho).$$

In total this shows

$$\partial_t = \cosh(\rho)\partial_\tau - \tau^{-1}\sinh(\rho)\partial_\rho. \tag{A.2.16}$$

Then we come to

$$Q(\partial_t,\partial_\tau) = \cosh(\rho)Q(\partial_\tau,\partial_\tau) - \tau^{-1}\sinh(\rho)Q(\partial_\rho,\partial_\tau).$$

Since $\partial_\tau, \partial_\rho$ are $\mathfrak{m}$-orthogonal, we immediately find $Q(\partial_\rho,\partial_\tau) = \partial_\rho\phi\partial_\tau\phi$. It now suffices to compute $Q(\partial_\tau,\partial_\tau)$, which we do by using (A.2.4):

$$\begin{aligned} Q(\partial_\tau,\partial_\tau) &= (\partial_\tau\phi)^2 - \frac{1}{2}\mathfrak{m}(\partial_\tau,\partial_\tau)\mathfrak{m}^{-1}(\mathrm{d}\phi,\mathrm{d}\phi) - \frac{1}{2}\mathfrak{m}(\partial_\tau,\partial_\tau)M^2\phi^2 \\ &= \frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2\tau^2}(\partial_\rho\phi)^2 + \frac{1}{2\tau^2\sinh^2(\rho)}|\partial_\theta\phi|^2 + \frac{1}{2}M^2\phi^2. \end{aligned}$$

We finally expose the coercivity by completing the square

$$
\begin{aligned}
Q(\partial_\tau,\partial_t) &= \frac{\cosh(\rho)}{2}(\partial_\tau\phi)^2 + \frac{\cosh(\rho)}{2\tau^2}(\partial_\rho\phi)^2 + \frac{\cosh(\rho)}{2\tau^2\sinh^2(\rho)}|\partial_\theta\phi|^2 \\
&+ \frac{\cosh(\rho)}{2}M^2\phi^2 - \tau^{-1}\sinh(\rho)\partial_\rho\phi\partial_\tau\phi \\
&= \frac{\cosh(\rho)}{2}\left(\partial_\tau\phi - \tau^{-1}\frac{\sinh(\rho)}{\cosh(\rho)}\partial_\rho\phi\right)^2 + \left(\frac{\cosh(\rho)}{2\tau^2} - \frac{\sinh^2(\rho)}{2\tau^2\cosh(\rho)}\right)(\partial_\rho\phi)^2 \\
&+ \frac{\cosh(\rho)}{2\tau^2\sinh^2(\rho)}|\partial_\theta\phi|^2 + \frac{\cosh(\rho)}{2}M^2\phi^2 \\
&= \frac{\cosh(\rho)}{2}\left(\partial_\tau\phi - \tau^{-1}\frac{\sinh(\rho)}{\cosh(\rho)}\partial_\rho\phi\right)^2 + \frac{1}{2\tau^2\cosh(\rho)}(\partial_\rho\phi)^2 \\
&+ \frac{\cosh(\rho)}{2\tau^2\sinh^2(\rho)}|\partial_\theta\phi|^2 + \frac{\cosh(\rho)}{2}M^2\phi^2.
\end{aligned}
$$

Finally, we recall (A.2.9, A.2.16) to write this as

$$
Q[\phi](\partial_\tau,\partial_t) = \frac{1}{2\tau^2\cosh(\rho)}\sum_{i=1}^{d}(L^i\phi)^2 + \frac{1}{2\cosh(\rho)}(\partial_t\phi)^2 + \frac{\cosh(\rho)}{2}M^2\phi^2. \tag{A.2.17}
$$

Integrating over Σ_τ concludes the proof of (2.2.12).

The only non-trivial estimates in (2.2.13) are the ones concerning $\|\phi\|_{\mathcal{L}^2_{-1}}$ and $\|\phi_t\|_{\mathcal{W}^{k,2}_{-1}}$. The estimate for ϕ follows from Hardy's inequality proved in Lemma A.2.5. We stress that the estimates for ϕ_t differ on both sides of the inequality (2.2.12) because on the left hand side we have integrals of the form

$$
\int_{\Sigma_\tau} \frac{1}{\cosh(\rho)}|\partial_t L^\alpha\phi|^2 \ \mathrm{dvol}_\tau,
$$

where on the right hand side we have integrals of the form

$$
\int_{\Sigma_\tau} \frac{1}{\cosh(\rho)}|L^\alpha\partial_t\phi|^2 \ \mathrm{dvol}_\tau.
$$

To show that these two are comparable, we induct the commutation relation $[L^i,\partial_t] = -\partial_{x^i}$, the algebraic relation $\partial_{x^i} = \frac{1}{t}L^i - \frac{x^i}{t}\partial_t$, and the fact that $x^i/t \approx 1$ in the interior light cone to estimate

$$
|L^\alpha\partial_t\phi| \lesssim \sum_{|\beta|\le|\alpha|}|\partial_t L^\beta\phi| + \sum_{|\beta|\le|\alpha|-1}\sum_{i=1}^{d}\frac{1}{\tau\cosh(\rho)}|L^iL^\beta\phi|.
$$

In the above inequality, $L^i L^\beta \phi \stackrel{\text{def}}{=} L^i \phi$ when $|\alpha| = 1$. The terms in the second sum can be controlled by

$$\frac{1}{\tau}|L^i L^\beta \phi|$$

because $\cosh(\rho) \geq 1$. Putting these estimates together we find that

$$\sum_{|\alpha|\leq\lfloor\frac{d}{2}\rfloor+1} \int_{\Sigma_\tau} \frac{1}{\cosh(\rho)}|L^\alpha \partial_t \phi|^2 \ \mathrm{dvol}_\tau \lesssim \sum_{|\alpha|\leq\lfloor\frac{d}{2}\rfloor+1} \int_{\Sigma_\tau} \frac{1}{\cosh(\rho)}|\partial_t L^\alpha \phi|^2 + \frac{1}{\tau^2\cosh(\rho)}\sum_{i=1}^{d}|L^i L^\alpha \phi|^2 \ \mathrm{dvol}_\tau. \quad \text{(A.2.18)}$$

By (A.2.17), the right hand side is bounded by

$$\sum_{|\alpha|\leq\lfloor\frac{d}{2}\rfloor+1} \mathcal{E}_\tau[L^\alpha \phi]^2,$$

as desired

□

A.3 Various computations for the perturbed system (4.2.17)

A.3.1 Computations supporting Section 4.5.1

A.3.1.1 Verification of (4.5.1)

$$\begin{aligned}
\Box_g \psi &= \frac{1}{\sqrt{|g|}}\partial_\mu\Big(\sqrt{|g|}g^{\mu\nu}\partial_\nu\psi\Big) \\
&= \frac{1}{\sqrt{|g|}}\partial_\mu\Big(\sqrt{|g|}\mathring{g}^{\mu\nu}\partial_\nu\psi\Big) - \frac{1}{\sqrt{|g|}}\partial_\mu\Big(\frac{1}{\sqrt{|g|}}\mathring{g}^{\mu\nu}\partial_\nu\phi\cdot\mathring{g}(\mathrm{d}\phi,\mathrm{d}\psi)\Big) \\
&= \frac{1}{2|g|}\mathring{g}(\mathrm{d}|g|,\mathrm{d}\psi) + \Box_{\mathfrak{m}}\psi + 2\partial_{\underline{u}}(\phi\Upsilon''\partial_{\underline{u}}\psi) \\
&\quad - \frac{1}{|g|}\cdot\underbrace{\sqrt{|g|}\partial_\mu\Big(\frac{1}{\sqrt{|g|}}\mathring{g}^{\mu\nu}\partial_\nu\phi\Big)}_{=\Upsilon''(\phi_{\underline{u}})^2}\cdot\mathring{g}(\mathrm{d}\phi,\mathrm{d}\psi) - \frac{1}{|g|}\mathring{g}\Big(\mathrm{d}\phi,\mathrm{d}\Big(\mathring{g}(\mathrm{d}\phi,\mathrm{d}\psi)\Big)\Big)
\end{aligned}$$

A.3.1.2 Verification of (4.5.2)

We start with

$$\mathcal{L}_X\Big(\sqrt{|g|}\partial_\mu \frac{\mathring{g}^{\mu\nu}\partial_\nu\phi}{\sqrt{|g|}}\Big) = X\Big(\Upsilon''(\phi_{\underline{u}})^2\Big).$$

Now, the left hand side can be written as

$$\begin{aligned}
&[X,\Box_{\mathfrak{m}}]\phi + \boxed{\Box_{\mathfrak{m}} X\phi} + 2[X,\partial_{\underline{u}}](\phi\Upsilon''\partial_{\underline{u}}\phi) \\
&\quad + 2\partial_{\underline{u}}\Big(X(\phi\Upsilon'')\partial_{\underline{u}}\phi\Big) + 2\partial_{\underline{u}}\Big(\phi\Upsilon''[X,\partial_{\underline{u}}]\phi\Big) + \boxed{2\partial_{\underline{u}}(\phi\Upsilon''\partial_{\underline{u}}X\phi)} \\
&\quad + \frac{1}{2|g|^2}X(|g|)\mathring{g}(\mathrm{d}\phi,\mathrm{d}|g|) - \frac{1}{2|g|}\mathcal{L}_X(\mathring{g}^{-1})(\mathrm{d}\phi,\mathrm{d}|g|) - \frac{1}{2|g|}\mathring{g}(\mathrm{d}X\phi,\mathrm{d}|g|) \\
&\quad - \frac{1}{2|g|}\mathring{g}\Big(\mathrm{d}\phi,\mathrm{d}\Big(\mathcal{L}_X(\mathring{g}^{-1})(\mathrm{d}\phi,\mathrm{d}\phi)\Big)\Big) - \boxed{\frac{1}{|g|}\mathring{g}\Big(\mathrm{d}\phi,\mathrm{d}\mathring{g}(\mathrm{d}\phi,\mathrm{d}X\phi)\Big)}.
\end{aligned}$$

Throughout we have used the Leibniz rule for Lie differentiation with respect to tensor contractions, as well as the fact that Lie derivatives commute with exterior differentiation. The boxed terms, we notice, are identical to the principal terms in $\Box_g\psi$ if we set $\psi = X\phi$. The formula (4.5.2) follows by rearranging the terms.

A.3.1.3 Control of $TT\phi$ terms

As the null structure that we require can all be recovered as discussed in Remark 4.5.3, for the control of the $TT\phi$ terms in terms of other $\mathscr{B}^{2,1}_*$ terms we do not need to be too precise with the weights. Starting from the equation

$$\Box_{\mathfrak{m}}\phi + 2\Upsilon''\partial_{\underline{u}}(\phi\phi_{\underline{u}}) - \frac{1}{2|g|}\mathring{g}(\mathrm{d}\phi,\mathrm{d}|g|) = \Upsilon''(\phi_{\underline{u}})^2$$

we first observe

$$\Box_{\mathfrak{m}}\phi = -\frac{\tau^2}{(y^0)^2}TT\phi \underbrace{- \frac{d}{y^0}T\phi + \frac{1}{(y^0)^2}\sum_{i=1}^{d} L^iL^i\phi - y^iL^iT\phi - y^iTL^i\phi}_{=\mathscr{B}^{2,1}_2\phi}.$$

Additionally, the quadratic terms

$$2\Upsilon''\partial_{\underline{u}}(\phi\phi_{\underline{u}}) - \Upsilon''(\phi_{\underline{u}})^2 = \mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)^2 + \phi\mathscr{P}_0\mathscr{W}_2(L^1L^1\phi + L^1T\phi + TL^1\phi + TT\phi + L^1\phi + T\phi).$$

The cubic and higher order terms are captured schematically by

$$\begin{aligned}\mathring{g}(\mathrm{d}\phi, \mathrm{d}|g|) = \mathscr{W}_2[\tau^2(\mathscr{B}_1^{1,1}\phi)^2 + (\mathscr{B}_0^{1,0}\phi)^2 + \phi\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)(L^1\phi + T\phi)]TT\phi \\ + [(\mathscr{B}_1^{1,1}\phi)^2 + \phi\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^2]\mathscr{B}_2^{2,1}\phi \\ + \mathscr{W}_1(\mathscr{B}_1^{1,1}\phi)^3 + \mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^4 + \phi\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^3 + \phi\mathscr{P}_1\mathscr{W}_1(\mathscr{B}_1^{1,1}\phi)^3,\end{aligned} \quad \text{(A.3.1)}$$

where we took care to isolate the terms with $TT\phi$ from other second derivatives.

This means that we can re-write

$$\begin{aligned}TT\phi = \frac{1}{\mathfrak{c}_{TT}}\Big(\mathscr{G}\mathscr{B}_2^{2,1}\phi + \mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)^2 + \mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^4 \\ + \phi\mathscr{P}_0\mathscr{W}_1(\mathscr{B}_1^{1,1}\phi) + \mathscr{G}(\mathscr{B}_1^{1,1}\phi)^3 + (1 + \phi\mathscr{P}_1)\mathscr{W}_1(\mathscr{B}_1^{1,1}\phi)^3\Big)\end{aligned} \quad \text{(A.3.2)}$$

where

$$\begin{aligned}\mathfrak{c}_{TT} \stackrel{\text{def}}{=} \frac{\tau^2}{(y^0)^2}\Big[1 + (\mathscr{B}_1^{1,1}\phi)^2 + \frac{1}{\tau^2}(\mathscr{B}_0^{1,0}\phi)^2 \\ + \frac{1}{\tau^2}(\mathscr{B}_0^{0,0}\phi)\mathscr{P}_0(1 + (\mathscr{B}_1^{1,1}\phi)(L^1\phi + T\phi))\Big].\end{aligned} \quad \text{(A.3.3)}$$

Remark A.3.1. Note that none of the $\mathscr{G}$ factors in (A.3.2) include any $\mathscr{B}_0^{1,0}\phi$ dependence.

A.3.1.4 Verification of (4.5.11)

We note the following very rough estimate for the cubic terms

$$\mathfrak{m}^{-1}(\mathrm{d}\psi_1, \mathrm{d}\psi_2) = \mathscr{B}_1^{1,1}\psi_1\mathscr{B}_1^{1,1}\psi_2$$

and hence

$$\mathring{g}(\mathrm{d}\psi_1, \mathrm{d}\psi_2) = \mathscr{B}_1^{1,1}\psi_1\mathscr{B}_1^{1,1}\psi_2(1 + \phi\mathscr{P}_0).$$

We also have that

$$\mathcal{L}_{L^1}(\mathring{g}^{-1})(\mathrm{d}\psi_1, \mathrm{d}\psi_2) = \mathscr{B}_0^{1,0}\phi\mathscr{P}_0\mathscr{B}_1^{1,1}\psi_1\mathscr{B}_1^{1,1}\psi_2. \tag{A.3.4}$$

So all the higher-order, non-boxed terms in (4.5.2) can be captured by the sum

$$\begin{aligned}\mathrm{HO}_1 \stackrel{\text{def}}{=} \mathscr{G}(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{1,1}L^1\phi)(\mathscr{B}_2^{2,2}\phi) + \mathscr{G}\mathscr{P}_0(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_2^{2,2}\phi)\\ + \mathscr{G}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^3(\mathscr{B}_1^{2,1}\phi) + \mathscr{G}\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^5(\mathscr{B}_2^{2,2}\phi) + \mathscr{G}\mathscr{P}_0(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^4\\ + \mathscr{G}\phi\mathscr{W}_1\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{2,1}\phi) + \mathscr{G}\phi^2\mathscr{W}_2\mathscr{P}_2(\mathscr{B}_1^{1,1}\phi)^4(\mathscr{B}_1^{2,1}\phi)\\ + \mathscr{G}\phi\mathscr{W}_1\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^5(\mathscr{B}_2^{2,2}\phi) + \mathscr{G}\mathscr{W}_1\mathscr{P}_1(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^3.\end{aligned} \tag{A.3.5}$$

Remark A.3.2. The term $\mathscr{G}(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{1,1}L^1\phi)(\mathscr{B}_2^{2,2}\phi)$ stands out in the expression of HO_1: it is both the only cubic term (all other terms are at least quartic in the unknowns) and the only term that is not explicitly multiplied by a factor of $\mathscr{P}_*$. In fact, this term is the only nonlinearity that would remain when $\Upsilon \equiv 0$, where the equations reduce to the small-data scenario studied by Lindblad [Lin04], and the nonlinearity is of the *double null* form $\mathfrak{m}^{-1}(\mathrm{d}\phi, \mathrm{d}(\mathfrak{m}^{-1}(\mathrm{d}\phi, \mathrm{d}\phi)))$.

We note that instead of writing $\mathscr{B}_1^{2,1}\phi$ we have chosen to write $\mathscr{B}_1^{1,1}L^1\phi$. This is deliberate in order to allow us to exploit certain improvements of decay for the $L^1\phi$ derivatives.

We concentrate on the boxed, quadratic terms in (4.5.2) next. For these terms we need the additional null structure as seen in (4.4.4), and we write, noting that $[L^1, \partial_{\underline{u}}] = -\partial_{\underline{u}}$, the following schematic decompositions for the quadratic terms:

$$\begin{aligned}
L^1(\Upsilon''(\phi_{\underline{u}})^2) &= \mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)(L^1L^1\phi + TL^1\phi) + \mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)^2,\\
\partial_{\underline{u}}(\phi\Upsilon''\phi_{\underline{u}}) &= \mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)^2\\
&\quad + \mathscr{P}_0\phi\mathscr{W}_2(L^1\phi + T\phi) + \mathscr{P}_0\phi\mathscr{W}_2(L^1L^1\phi + TL^1\phi + TT\phi),\\
\partial_{\underline{u}}(L^1(\phi\Upsilon'')\phi_{\underline{u}}) &= \mathscr{P}_0(\phi + L^1\phi)\mathscr{W}_2(L^1\phi + T\phi)\\
&\quad + \mathscr{P}_0(\phi + L^1\phi)\mathscr{W}_2(L^1L^1\phi + TL^1\phi + TT\phi)\\
&\quad + \mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)^2 + \mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)(L^1L^1\phi + TL^1\phi).
\end{aligned}$$

So we can summarize the quadratic nonlinearities schematically as

$$\begin{aligned}
\mathrm{QN}_1 = \mathscr{P}_0\mathscr{W}_2\cdot\Big[&(L^1\phi + T\phi)^2 + T\phi(L^1L^1\phi + TL^1\phi)\\
&+ \phi(L^1\phi + T\phi) + (\phi + L^1\phi)(L^1L^1\phi + TL^1\phi + TT\phi)\Big]. \quad \text{(A.3.6)}
\end{aligned}$$

A.3.1.5 Verification of (4.5.12)

In the case where $X = L^i$ for $i \neq 1$, we have that

$$\mathcal{L}_{L^i}(\mathring{g}^{-1})(\mathrm{d}\psi_1, \mathrm{d}\psi_2) = \mathscr{B}_0^{1,0}\phi\mathscr{P}_1\mathscr{B}_1^{1,1}\psi_1\mathscr{B}_1^{1,1}\psi_2. \quad \text{(A.3.7)}$$

One can check that the higher-order, non-boxed terms in (4.5.2) are now captured by

$$\begin{aligned}
\mathrm{HO}_i = \mathrm{HO}_1 &+ \mathscr{G}(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_2^{2,2}\phi)\\
&+ \mathscr{G}\phi\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_2^{2,2}\phi) + \mathscr{G}\phi\mathscr{W}_1\mathscr{P}_2(\mathscr{B}_1^{1,1}\phi)^3 + \mathscr{G}\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^4. \quad \text{(A.3.8)}
\end{aligned}$$

The added terms are now the pure cubic term which now can include L derivatives in all directions, and additional quartic terms which arises from X hitting Υ'' which generates a $\mathscr{P}_1$ instead of $\mathscr{P}_0$.

The quadratic parts of the nonlinearity can also be expanded schematically. The computations are as follows:

$$L^i(\Upsilon''(\phi_{\underline{u}})^2) = \mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)(L^iL^1\phi + L^iT\phi) + \mathscr{P}_1\mathscr{W}_2(L^1\phi + T\phi)^2,$$

$$\begin{aligned}\frac{1}{y^0}(L^i - y^iT)(\phi\Upsilon''\phi_{\underline{u}}) &= \mathscr{P}_0\mathscr{B}_1^{1,1}\phi\mathscr{W}_1(L^1\phi + T\phi) + \mathscr{P}_1\mathscr{W}_2\phi(L^1\phi + T\phi)\\ &\quad + \mathscr{P}_0\phi\mathscr{W}_1(\mathscr{B}_1^{1,1}L^1\phi + \mathscr{B}_1^{1,1}T\phi),\end{aligned}$$

$$\begin{aligned}\partial_{\underline{u}}(L^i(\phi\Upsilon'')\phi_{\underline{u}}) &= \mathscr{P}_0\mathscr{W}_2(L^1\phi + T\phi)(L^1L^i\phi + TL^i\phi) + \mathscr{P}_1\mathscr{W}_2(L^1\phi + T\phi)^2\\ &\quad + \mathscr{P}_0\mathscr{W}_2L^i\phi(L^1L^1\phi + TL^1\phi + TT\phi + L^1\phi + T\phi)\\ &\quad + \mathscr{P}_1\mathscr{W}_2\phi(L^1L^1\phi + TL^1\phi + TT\phi + L^1\phi + T\phi),\end{aligned}$$

$$\partial_{\underline{u}}(\phi\Upsilon''\frac{1}{y^0}(L^i - y^iT)\phi) = \mathscr{P}_0\mathscr{W}_1(L^1\phi + T\phi)\mathscr{B}_1^{1,1}\phi + \mathscr{P}_0\mathscr{W}_1\phi(L^1\mathscr{B}_1^{1,1}\phi + T\mathscr{B}_1^{1,1}\phi).$$

Thus we can collect the quadratic nonlinearities using the schematic expression

$$\begin{aligned}\mathrm{QN}_i = \mathscr{P}_0\mathscr{W}_1(\phi + L^1\phi + T\phi)(\mathscr{B}_1^{1,1}L^1\phi + \mathscr{B}_2^{2,2}\phi + \mathscr{B}_1^{1,1}\phi)&\\ + \mathscr{W}_2(\mathscr{P}_0L^i\phi + \mathscr{P}_1\phi)(L^1L^1\phi + TL^1\phi + TT\phi + L^1\phi + T\phi)&\\ + \mathscr{P}_1\mathscr{W}_2(\phi + L^1\phi + T\phi)(L^1\phi + T\phi).& \quad \text{(A.3.9)}\end{aligned}$$

A.3.2 Computations supporting 4.5.2

Observe that, expanding using the standard formula for the Laplace-Beltrami operator and (4.2.14), we obtain

$$\begin{aligned}\Box_g\psi = \Box_m\psi + \frac{1}{2}\frac{1}{|g|}g^{-1}(\mathrm{d}|g|, \mathrm{d}\psi) + 2\partial_{\underline{u}}(\phi\Upsilon''\partial_{\underline{u}}\psi)&\\ - \frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}\phi, \mathrm{d}(\frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}\phi, \mathrm{d}\psi))) - \frac{1}{|g|}\Upsilon''(\phi_{\underline{u}})^2\mathring{g}^{-1}(\mathrm{d}\phi, \mathrm{d}\psi).&\end{aligned}$$

This implies

$$
\begin{aligned}
[X,\Box_g]\psi &= \frac{1}{2}g^{-1}(\mathrm{d}(X\ln|g|),\mathrm{d}\psi)+\frac{1}{2}\mathcal{L}_X g^{-1}(\mathrm{d}\ln|g|,\mathrm{d}\psi)\\
&\quad+\boxed{2[X,\partial_{\underline{u}}](\phi\Upsilon''\partial_{\underline{u}}\psi)}+\boxed{2\partial_{\underline{u}}(X(\phi\Upsilon'')\partial_{\underline{u}}\psi)}+\boxed{2\partial_{\underline{u}}(\phi\Upsilon''[X,\partial_{\underline{u}}]\psi)}\\
&\quad+\frac{1}{2}|g|^{-2/3}X(|g|)\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}(\frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi)))-\frac{1}{\sqrt{|g|}}\mathcal{L}_X(\mathring{g}^{-1})(\mathrm{d}\phi,\mathrm{d}(\frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi)))\\
&\quad-\frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}(X\phi),\mathrm{d}(\frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi)))+\frac{1}{2}\frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}(\frac{X|g|}{|g|^{3/2}}\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi)))\\
&\quad-\frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}(\frac{1}{\sqrt{|g|}}\mathcal{L}_X(\mathring{g}^{-1})(\mathrm{d}\phi,\mathrm{d}\psi)))-\frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}(\frac{1}{\sqrt{|g|}}\mathring{g}^{-1}(\mathrm{d}(X\phi),\mathrm{d}\psi)))\\
&\quad-X(\frac{1}{|g|}\Upsilon''(\phi_{\underline{u}})^2)\mathring{g}^{-1}(\mathrm{d}\phi,\mathrm{d}\psi)-\frac{1}{|g|}\Upsilon''(\phi_{\underline{u}})^2\mathcal{L}_X(\mathring{g}^{-1})(\mathrm{d}\phi,\mathrm{d}\psi)\\
&\quad-\frac{1}{|g|}\Upsilon''(\phi_{\underline{u}})^2\mathring{g}^{-1}(\mathrm{d}(X\phi),\mathrm{d}\psi).
\end{aligned}
$$

Except for the three boxed terms, which are linear in both ϕ and ψ, all remaining terms are at least quadratic in ϕ.

The terms that are quadratic and above in ϕ are generally harmless. We will use the following rough estimate for the quadratic form g^{-1}:

$$
g^{-1}(\mathrm{d}\psi_1,\mathrm{d}\psi_2)=\mathscr{B}_1^{1,1}\psi_1\mathscr{B}_1^{1,1}\psi_2(1+\phi\mathscr{P}_0+\mathscr{G}(\mathscr{B}_1^{1,1}\phi)^2).
$$

For $X=L^i$, where $i=1,\dots,3$, the terms quadratic and above in ϕ can be schematically captured by the following collection of terms:

$$
\begin{aligned}
&(X\mathscr{G})\cdot\Big[(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_2^{2,2}\phi)(\mathscr{B}_1^{1,1}\psi)+(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_2^{2,2}\psi)\\
&\qquad+\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^3(\mathscr{B}_1^{1,1}\psi)+\mathscr{P}_1\mathscr{W}_1\phi(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{1,1}\psi)\Big]\\
&\quad+\mathscr{G}\cdot\Big[(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_2^{2,2}\phi)(\mathscr{B}_1^{1,1}\psi)+(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_2^{3,2}\phi)(\mathscr{B}_1^{1,1}\psi)\\
&\quad+(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_2^{2,2}\psi)+\mathscr{P}_0(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_1^{1,1}\psi)\\
&\quad+\mathscr{P}_1(\mathscr{B}_1^{1,1}\phi)^3(\mathscr{B}_1^{1,1}\psi)+\mathscr{P}_1\mathscr{W}_1\phi(\mathscr{B}_1^{1,1}\phi)(\mathscr{B}_1^{2,1}\phi)(\mathscr{B}_1^{1,1}\psi)\\
&\quad+\mathscr{P}_1\mathscr{W}_1(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{1,1}\psi)+\mathscr{P}_2\mathscr{W}_1\phi(\mathscr{B}_1^{1,1}\phi)^2(\mathscr{B}_1^{1,1}\phi)\Big]
\end{aligned}
\tag{A.3.10}
$$

where since $X=L^i$ we have that

$$
X\mathscr{G}=\mathscr{G}\cdot\Big(\mathscr{B}_0^{1,0}\phi\mathscr{P}_0+\phi\mathscr{P}_1+\mathscr{B}_0^{2,0}\phi+\mathscr{B}_1^{2,1}\phi\Big). \tag{A.3.11}
$$

We note, as before, all terms involving $\mathscr{P}_*$ weights are at least cubic in ϕ.

The terms that are linear in ϕ in the commutator can also be expanded. For higher level commutations we do not need to separate between L^1 and L^i for $i \neq 1$. So we can just write $[X, \partial_{\underline{u}}] = \mathscr{B}_1^{1,1}$, which allows us to capture the relevant terms by

$$\begin{aligned}
[X, \partial_{\underline{u}}](\phi\Upsilon''\psi_{\underline{u}}) &= \mathscr{P}_0\mathscr{B}_1^{1,1}\phi\mathscr{W}_1(L^1\psi + T\psi) + \mathscr{P}_1\mathscr{W}_2\phi(L^1\psi + T\psi) \\
&\quad + \mathscr{P}_0\phi\mathscr{W}_1\mathscr{B}_1^{1,1}(L^1\psi + T\psi), \\
\partial_{\underline{u}}(X(\phi\Upsilon'')\psi_{\underline{u}}) &= (\mathscr{P}_1\phi + \mathscr{P}_0\mathscr{B}_0^{1,0}\phi)\mathscr{W}_2(L^1\psi + T\psi + L^1L^1\psi + TL^1\psi + TT\psi) \\
&\quad + \mathscr{P}_0\mathscr{B}_1^{1,1}\phi\mathscr{W}_1(L^1\psi + T\psi) + \mathscr{P}_0\mathscr{W}_2\mathscr{B}_0^{1,0}(L^1\phi + T\phi)(L^1\psi + T\psi) \\
&\quad + \mathscr{P}_1\mathscr{W}_2(L^1\phi + T\phi)(L^1\psi + T\psi) + \mathscr{P}_1\mathscr{W}_2\phi(L^1\psi + T\psi), \\
\partial_{\underline{u}}(\phi\Upsilon''[X, \partial_{\underline{u}}]\psi) &= \mathscr{P}_0\mathscr{W}_1(L^1\phi + T\phi)\mathscr{B}_1^{1,1}\psi + \mathscr{P}_0\mathscr{W}_1\phi(L^1\mathscr{B}_1^{1,1}\psi + T\mathscr{B}_1^{1,1}\psi).
\end{aligned}$$

Using the commutation relations of Proposition 4.3.12 we can summarize these terms by

$$\begin{aligned}
&\mathscr{P}_0\mathscr{W}_1(\mathscr{B}_1^{1,1}\phi)(L^1\psi + T\psi) + \mathscr{P}_0\mathscr{W}_1(\phi + L^1\phi + T\phi)(\mathscr{B}_1^{1,1}\psi) \\
&\quad + \mathscr{P}_0\mathscr{W}_1(\mathscr{B}_0^{1,0}\phi)(\mathscr{B}_1^{1,1}L^1\psi + \mathscr{B}_1^{1,1}T\psi) + \mathscr{P}_0\mathscr{W}_2(\mathscr{B}_0^{1,0}L^1\phi + \mathscr{B}_0^{1,0}T\phi)(L^1\psi + T\psi) \\
&\qquad + \mathscr{P}_1\mathscr{W}_2(\phi + L^1\phi + T\phi)(L^1\psi + T\psi) + \mathscr{P}_1\mathscr{W}_2\phi(L^1L^1\psi + TL^1\psi + TT\psi) \qquad \text{(A.3.12)}
\end{aligned}$$

which have similar structure to the terms appearing in QN_1 and QN_i above.

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